The spectral dimension of random brushes

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Abstract. We consider a class of random graphs, called random brushes, which are constructed by adding linear graphs of random lengths to the vertices of $\mathbb{Z}^d$ viewed as a graph. We prove that for $d = 2$ all random brushes have spectral dimension $d_s = 2$. For $d = 3$ we have $\frac{5}{2} \leq d_s \leq 3$ and for $d \geq 4$ we have $3 \leq d_s \leq d$.

1 Introduction

The generic structure of random geometrical objects is of interest in many branches of physics ranging from condensed matter physics to quantum gravity, see e.g. [1] and [2]. One of the methods used to analyze such objects is to study diffusion or random walk. Diffusion allows us to define a notion of dimension, the spectral dimension, for random geometrical objects. In recent years the spectral dimension of triangulations has been studied numerically in quantum gravity [5, 6, 7, 8, 9] and analytically for certain classes of random trees [10, 3, 4]. In [3] the spectral dimension of various ensembles of random combs was calculated. In this article we generalize the monotonicity results of [3] which allows us to find bounds on the spectral dimensions of a class of graphs which we call brushes and define below.

Let G be a connected, locally finite (i.e. each vertex has finitely many nearest neighbours) rooted graph. All graphs that we consider will be assumed to have this property. Let $p_G(t)$ be the probability that a simple random walk on G which starts at the root is back at the root after t steps. If

$$p_G(t) \sim t^{-d_s/2} \tag{1}$$

as $t \to \infty$ then we say that d_s is the spectral dimension of the graph G. The existence of d_s is not guaranteed for individual graphs but its ensemble average can be shown to be well defined in many cases [3, 4]. It is easy to see that if the spectral dimension exists then it is independent of the starting site of the random walk.

Let us view $\mathbb{Z}^d$ as a graph with $j, k \in \mathbb{Z}^d$ neighbours if their distance is 1 and let the origin of $\mathbb{Z}^d$ be the root. It is well known that the spectral dimension of $\mathbb{Z}^d$ is d. Let N_l be a linear chain of length ℓ, i.e., the graph obtained be connecting nearest neighbours in $\{0, 1, \ldots, \ell\}$ with a link. Let 0 be the root of N_ℓ. Similarly, let N_∞ be the infinite linear chain with root at 0. A d-brush is a graph constructed by attaching one of the graphs N_ℓ to each vertex of $\mathbb{Z}^d$ by identifying the root of N_ℓ with a vertex in $\mathbb{Z}^d$, $\ell \in \mathbb{N}_0 \cup \{\infty\}$, $\ell = 0$ corresponding to the empty chain. In a brush B we will refer to $\mathbb{Z}^d$ as the base and the linear chains as bristles. A random brush is defined by letting the length of the bristles be identically and independently distributed by a probability measure on $\mathbb{N}_0 \cup \{\infty\}$. We see that the case $d = 1$ corresponds to the combs studied in [3] which were shown to have a spectral dimension in the interval $[1, \frac{3}{2}]$.

For $d > 1$ we will show that the spectral dimensions of random brushes satisfy the following:

$$\begin{aligned} d_s &= 2, \quad \text{if} \quad d = 2, \\ \frac{5}{2} \leq \quad d_s &\leq 3, \quad \text{if} \quad d = 3, \\ 3 \leq \quad d_s &\leq d, \quad \text{if} \quad d \geq 4. \end{aligned} \tag{2}$$

Some comments are in order. We see that when $d \geq 3$, attaching the bristles to the base serves to lower the spectral dimension since the spectral dimension of $\mathbb{Z}^d$ is equal to d. This is opposite to the case of combs where the linear chains tended to increase the spectral dimension. Intuitively this can be understood in the following way. If there is a very long bristle somewhere, a random walk can go up it and spend a long time there before returning to the base which it must do eventually since the bristles are recurrent. Once it returns to the base it will go back to the root with nonzero probability. We will indeed see below that adding a single infinite bristle to $\mathbb{Z}^d$ with $d \geq 4$ will bring the spectral dimension down to 3. The two dimensional case is special because $\mathbb{Z}^2$ is only marginally recurrent and the generating function for $p_{\mathbb{Z}^2}(t)$ has a logarithmic singularity which is not changed by the presence of bristles. Assuming that the spectral dimension of random brushes can be calculated by mean field theory we show that the full range of exponents in (2) is realized.

The paper is organized as follows. In the next section we define the generating functions used to analyze the spectral dimension. We then establish generalized monotonicity lemmas which are shown to imply the stated bounds on d_s in Section 4. Section 5 contains a discussion of mean field theory for brushes. A final section contains some comments.

2 Generating Functions

Let G be a graph and $p_G^1(t)$ the probability that a random walk is at the root at time t for the first time after $t = 0$. We define the return generating function

$$Q_G(z) = \sum_{t=0}^{\infty} p_G(t) z^t \tag{3}$$

and the first return generating function

$$P_G(z) = \sum_{t=0}^{\infty} p_G^1(t) z^t. \tag{4}$$

The generating functions are related by

$$Q_G(z) = \frac{1}{1 - P_G(z)}. \tag{5}$$

If G has a spectral dimension d_s then

$$Q_G^{(n)}(z) \sim \begin{cases} 1 & \text{if } n = d_s/2 - 1 \\ (1-z)^{d_s/2-1-n} & \text{otherwise} \end{cases} \tag{6}$$

where n is the smallest nonnegative integer for which $Q_G^{(n)}(z)$ diverges as $z \to 1$. Similarly, the behaviour (6) implies that the spectral dimension is d_s. Here $f(y) \sim y^\alpha$ as $y \to 0$ means that for any $\epsilon > 0$ there exist positive constants c_1 and c_2, which may depend on ϵ, such that

$$c_1 y^{\alpha+\epsilon} \leq f(y) \leq c_2 y^{\alpha-\epsilon} \tag{7}$$

for y small enough. Note that $f(y) \sim 1$ allows f to have a logarithmic singularity at 0.

The function $P_G(z)$ is analytic in the unit disc and $|P(z)| < 1$ for $|z| < 1$. If $P_G(z) \to 1$ as $z \to 1$ then $Q_G(z)$ clearly diverges in which case the random walk is recurrent and $d_s \leq 2$. If $P_G(z) \not\to 1$ as $z \to 1$ then the random walk is transient and $d_s \geq 2$. In the latter case we see that if some derivative $Q^{(n)}(z)$ diverges as $z \to 1$ then $Q_G^{(n)}(z) \sim P_G^{(n)}(z)$ as $z \to 1$.

If a graph has the property that every random walk which begins and ends at the root has an even number of steps, as is the case for brushes and bristles, we have to replace $p_G(t)$ with $p_G(2t)$ in (1) and z with z^2 on the right hand side of (6). Then it is convenient to introduce a variable $x = 1 - z^2 \in [0, 1]$. We will use the variable z for general graphs but the variable x when dealing with brushes and bristles.

We will need the following first return generating functions for the graphs N_l and N_∞ [3]

$$P_l(x) = 1 - \sqrt{x}\frac{(1+\sqrt{x})^l - (1-\sqrt{x})^l}{(1+\sqrt{x})^l + (1-\sqrt{x})^l} \tag{8}$$

and

$$P_\infty(x) = 1 - \sqrt{x}. \tag{9}$$

Let μ be a probability measure on $\mathbb{N}_0 \cup \{\infty\}$. Let $\mathcal{B}^d$ be the set of all d-brushes. We define a probability measure π on $\mathcal{B}^d$ by letting the measure of the set of d-brushes Ω which have bristles at $n_1, n_2, ..., n_k \in \mathbb{Z}^d$ of length $\ell_1, \ell_2, ..., \ell_k$ be

$$\pi(\Omega) = \prod_{i=1}^{k} \mu(l_i). \tag{10}$$

The set $\mathcal{B}^d$ together with π defines a random brush ensemble. We define the averaged generating functions

$$\overline{P}(x) = \langle P_B(x) \rangle_\pi \tag{11}$$

and

$$\overline{Q}(x) = \langle Q_B(x) \rangle_\pi \tag{12}$$

where $\langle \cdot \rangle_\pi$ denotes expectation with respect to π. We say that a random brush has spectral dimension d_s if $\overline{Q}(x)$ obeys the relation (6) (after replacing z with z^2 on the right hand side).

3 Monotonicity

Here we present the monotonicity results in a slightly more general setting than is needed for the applications below. This is both for clarity and potential applications to random graphs different from the brushes.

Let G_1 and G_2 be graphs such that G_1 can be constructed from G_2 by attaching rooted graphs $F(i)$ by their roots to sites $i \neq r$ of G_2. Let the roots of G_1 and G_2 be the same vertex (regarding G_2 as a subgraph of G_1). The following result is a generalization of the Monotonicity Lemma of [3].

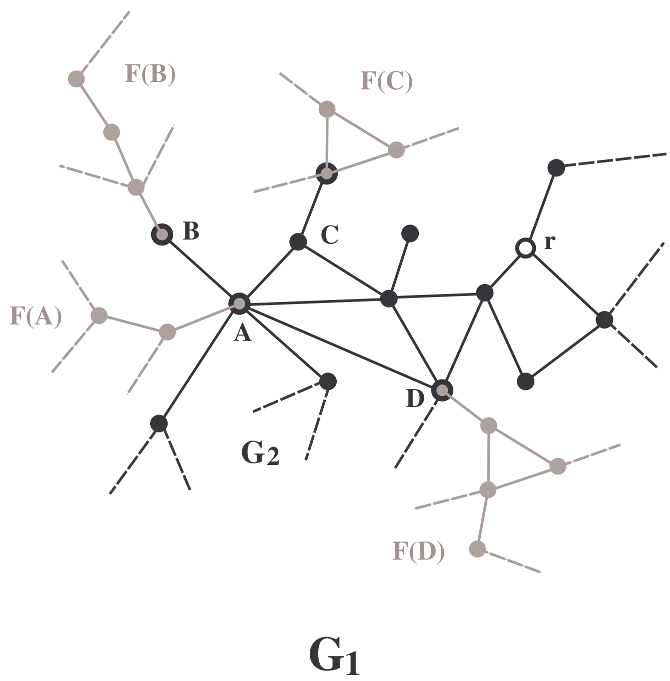

Figure 1: An example of a graph G_1 constructed from G_2 and the $F(i)$'s.

Lemma 1 *With G_1 and G_2 defined as above and $G_1 \neq G_2$ we have*

$$P_{G_1}(z) \leq P_{G_2}(z) \tag{13}$$

with equality if and only if all the $F(i)$'s are recurrent and z = 1.

Proof: For any graph G we can write $P_G(z)$ as a weighted sum over all random walks ω on G which start and end at the root without intermediate visits to the root (this condition is denoted 'ω: FR on G'). Each walk ω has a weight

$$W_G(\omega) = \prod_{t=0}^{|\omega|-1} \sigma_G(\omega_t)^{-1} \tag{14}$$

where $\sigma_G(\omega_t)$ is the order of the vertex ω_t on G where the walk is located at time t and $|\omega|$ is the number of steps in ω. Each step of a walk has a factor z associated with it so

$$P_G(z) = \sum_{\omega\text{: FR on } G} W_G(\omega) z^{|\omega|}. \tag{15}$$

Now consider a random walk ω' on G_1 which starts at the root. Let ω be the subwalk of ω' which only travels on G_2. If we look at the walk ω at time t and location ω_t then ω can be a subwalk of many different walks ω' which correspond to all possible excursions into the graph $F(\omega_t)$ before returning back to the walk on G_2. The weight of these excursions is

$$\sum_{n=0}^{\infty}\Big(\frac{\sigma_{F(\omega_t)}(\omega_t)}{\sigma_{G_1}(\omega_t)}P_{F(\omega_t)}(z)\Big)^n = \frac{1}{1-\Big(\frac{\sigma_{F(\omega_t)}(\omega_t)}{\sigma_{G_1}(\omega_t)}P_{F(\omega_t)}(z)\Big)} \tag{16}$$

where n counts the number of visits to ω_t before the walk leaves ω_t for another vertex on G_2 and the factor in front of $P_{F(\omega_t)}(z)$ changes the order of the root of $F(\omega_t)$ to $\sigma_{G_1}(\omega_t) = \sigma_{G_2}(\omega_t) + \sigma_{F(\omega_t)}(\omega_t)$. The weight of the first step back into G_2 after all the visits to $F(\omega_t)$ is

$$\frac{z}{\sigma_{G_1}(\omega_t)}. \tag{17}$$

Now replace the original weight $\sigma_{G_2}(\omega_t)^{-1}z$ of ω at each point $\omega_t \neq \omega_0$ by the product of the factors (16) and (17). This newly weighted ω then accounts for every random walk on G_1 which has ω as a subwalk on G_2. Thus we can write

$$\begin{aligned} P_{G_1}(z) &= \sum_{\omega\text{: FR on } G_2} \sigma_{G_2}(\omega_0)^{-1} z \prod_{t=1}^{|\omega|-1}\Big(\frac{z}{\sigma_{G_2}(\omega_t)+\sigma_{F(\omega_t)}(\omega_t)(1-P_{F(\omega_t)}(z))}\Big) \\ &= \sum_{\omega\text{: FR on } G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) z^{|\omega|} \end{aligned} \tag{18}$$

where in the last step we defined

$$K_{G_1,G_2}(z;\omega) = \prod_{t=1}^{|\omega|-1}\Big(\frac{\sigma_{G_2}(\omega_t)}{\sigma_{G_2}(\omega_t)+\sigma_{F(\omega_t)}(\omega_t)(1-P_{F(\omega_t)}(z))}\Big). \tag{19}$$

Since $P_{F(\omega_t)}(z) \leq 1$ with equality if and only if $F(\omega_t)$ is recurrent and $z = 1$ it is clear that $K_{G_1,G_2}(z;\omega) \leq 1$ for all z with equality if and only if all the graphs $F(\omega_t)$ for a given ω on G_2 are recurrent and $z = 1$. The inequality (13) follows.

$\square$

Lemma 2 *Let $n \in \mathbb{Z}^+$ be such that $P_{G_2}^{(n-1)}(z)$ is continuous on the closed interval $[0,1]$. If all the $F(i)$'s are recurrent then for a given $z \in]0,1[$ there exists a $\xi \in]z,1[$ such that*

$$P_{G_1}^{(n)}(\xi) \geq P_{G_2}^{(n)}(\xi). \tag{20}$$

Proof: We define

$$H_{G_1,G_2}(z;n) = \sum_{\omega:\text{ FR on } G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \frac{d^{n-1}}{dz^{n-1}} z^{|\omega|} \tag{21}$$

where K_{G_1,G_2} is defined as above. Every derivative of a (first) return generating function is a positive increasing function of $z \in [0,1[$ since the power series have no negative coefficients. It is easy to verify that the function K_{G_1,G_2} has the same property. Therefore we get by differentiating (18) n times

$$\begin{aligned} P_{G_1}^{(n)}(z) &= \sum_{i=0}^{n} \binom{n}{i} \sum_{\omega:\text{ FR on } G_2} K_{G_1,G_2}^{(i)}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n-i)} \\ &\geq \sum_{\omega:\text{ FR on } G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n)} \\ &+ \quad n \sum_{\omega:\text{ FR on } G_2} K'_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n-1)} \\ &\geq \sum_{\omega:\text{ FR on } G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n)} \\ &+ \sum_{\omega:\text{ FR on } G_2} K'_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n-1)} \\ &= H'_{G_1,G_2}(z;n). \end{aligned} \tag{22}$$

With the same argument as in the proof of Lemma 1 it holds that $H_{G_1,G_2}(z;n) \leq P_{G_2}^{(n-1)}(z)$ with equality when $z = 1$ since all the $F(i)$'s are recurrent and because $P_{G_2}^{(n-1)}(z)$ and therefore also $H_{G_1,G_2}(z;n)$ are continuous on $[0,1]$. Since $H_{G_1,G_2}(z;n)$ and $P_{G_2}^{(n-1)}(z)$ are positive and increasing functions of z we find that

$$\frac{H_{G_1,G_2}(1;n) - H_{G_1,G_2}(z;n)}{P_{G_2}^{(n-1)}(1) - P_{G_2}^{(n-1)}(z)} \geq 1. \tag{23}$$

By a generalized mean-value theorem there exists a $\xi \in]z,1[$ such that

$$\frac{H_{G_1,G_2}(1;n) - H_{G_1,G_2}(z;n)}{P_{G_2}^{(n-1)}(1) - P_{G_2}^{(n-1)}(z)} = \frac{H'_{G_1,G_2}(\xi;n)}{P_{G_2}^{(n)}(\xi)}. \tag{24}$$

In view of (22) the Lemma follows.

□

Theorem 1 *Assume that all the $F(i)$'s are recurrent and that G_1 and G_2 have spectral dimensions d_{s_1} and d_{s_2} respectively. If G_2 is recurrent then G_1 is recurrent and $d_{s_1} \geq d_{s_2}$. If G_2 is transient then G_1 is transient and $d_{s_1} \leq d_{s_2}$.*

Proof: Since all the $F(i)$'s are recurrent we have $P_{G_1}(1) = P_{G_2}(1)$ and therefore if G_2 is transient/recurrent then so is G_1. First assume that G_2 is recurrent. Then by using Lemma 1 and Equations (5), (6) and (7) we find that for any $\epsilon > 0$ there exist positive constants c_1 and c_2 which may depend on ϵ such that

$$c_1(1-z)^{d_{s_1}/2-1+\epsilon} \leq Q_{G_1}(z) \leq Q_{G_2}(z) \leq c_2(1-z)^{d_{s_2}/2-1-\epsilon} \tag{25}$$

for z close to 1. If $d_{s_1} \neq d_{s_2}$ we choose $\epsilon < \frac{1}{4}|d_{s_2} - d_{s_1}|$ and send $z \to 1$ to conclude that $d_{s_1} > d_{s_2}$. When G_2 is transient we use Lemma 2 and similar arguments as above to show that $d_{s_1} \leq d_{s_2}$.

□

4 The Spectral Dimension

The d-brush where every bristle is N_∞ we call the full d-brush and denote it $*d$. We can relate the generating function of the full d-brush to the generating functions of $\mathbb{Z}^d$ and N_∞. We use the same argument as in the proof of Lemma 1. Replacing all the graphs $F(i)$ with N_∞ and noting that the order of every point in $\mathbb{Z}^d$ is $1/2d$ we get

$$P_{*d}(x) = \Big(1 + \frac{1-P_\infty(x)}{2d}\Big) P_{\mathbb{Z}^d}(x_{\text{ren}}(x)) \tag{26}$$

where x_{ren} is defined by

$$\sqrt{1-x_{\text{ren}}} = \frac{\sqrt{1-x}}{1+\frac{1-P_\infty(x)}{2d}}. \tag{27}$$

We see that $x_{\text{ren}} = \sqrt{x}/d + O(x)$. By differentiating (26) once and comparing with (6) we find the spectral dimension of the full brush

$$d_* = \begin{cases} \frac{d}{2}+1 & \text{if } 1 \leq d \leq 4 \\ 3 & \text{if } d \geq 4. \end{cases} \tag{28}$$

If we replace the infinite bristles with finite ones, all of which have the same length, then with the same calculation we see that the spectral dimension remains equal to d. These are special cases of a more general result obtained in [11] for so called bundled structures. There, the base $\mathbb{Z}^d$ can be replaced by any graph B and the infinite bristle (fiber) can also be replaced by any fixed graph F.

Using the above calculation and Theorem 1 we can find bounds on the spectral dimensions of fixed and random d-brushes. Any fixed d-brush B can be constructed from $\mathbb{Z}^d$ by attaching (recurrent) bristles to it and the full d-brush can be constructed from B by attaching (recurrent) bristles to it. Therefore, by Theorem 1, the spectral dimension of any fixed d-brush, if it exists, lies between d and d_*. This also holds for random brushes as is clear from equations (33) and (36) below and the proof of Theorem 1. The spectral dimension for any fixed or random d-brush, if it exists, therefore obeys the inequalities (2).

The spectral dimension of random 2-brushes always equals 2. Indeed it follows from the fact that $Q_{\mathbb{Z}^2}(x)$ is asymptotic to $|\ln(x)|$ as $x \to 0$ and Lemma 1 that there exist positive constants c_1 and c_2 such that

$$c_1|\ln(x)| \leq \overline{Q}(x) \leq c_2|\ln(x)| \tag{29}$$

when x is small enough . This is a stronger condition on the asymptotic behavior of $\overline{P}(x)$ than $\overline{P}(x) \sim 1$ as $x \to 0$.

It is interesting that for $d \geq 4$ the lower bound on the spectral dimension always equals 3. In fact it is easy to see that attaching a single infinite bristle to $\mathbb{Z}^d$ with $d \geq 4$ reduces the spectral dimension to 3. We can show this by attaching an infinite bristle to the root of $\mathbb{Z}^d$ since the spectral dimension is independent of the starting site of the random walks. Let us call the resulting brush $\perp d$. The first return generating function is simply

$$P_{\perp d}(x) = \frac{2d}{2d+1} P_{\mathbb{Z}^d}(x) + \frac{1}{2d+1} P_{\infty}(x). \tag{30}$$

Since $d \geq 4$ equation (6) shows that $Q'_{\mathbb{Z}^d}(x)$ diverges slower than any negative power of x as $x \to 0$ but $Q'_{\infty}(x) \sim x^{-1/2}$. Therefore by differentiating (30) we get

$$Q'_{\perp d}(x) \sim x^{-1/2} \tag{31}$$

as $x \to 0$ and therefore by (6) the spectral dimension equals 3. It follows that if a random d-brush with $d \geq 4$ has a nonzero probability of having one or more infinite bristles its spectral dimension equals 3.

We find with similar arguments that adding a single (or finitely many) infinite bristles to $\mathbb{Z}^3$ gives the spectral dimension 3. However, if we add infinitely many bristles the spectral dimension of $\mathbb{Z}^3$ can be lowered as is seen e.g. in the case of the full 3-brush.

We now use the notation of Section 3 and consider the case when $G_2 = \mathbb{Z}^d$ and instead of having a fixed G_1 we take a random d-brush. We would like to get bounds for the spectral dimension of random brushes similar to those in Theorem 1. First we note that by Lemma 1 we have for any $B \in \mathcal{B}^d$ that

$$P_{*d}(x) \leq P_B(x) \leq P_{\mathbb{Z}^d}(x) \tag{32}$$

and averaging we get

$$P_{*d}(x) \leq \overline{P}(x) \leq P_{\mathbb{Z}^d}(x). \tag{33}$$

In order to generalize Lemma 2 to random brushes we consider the case $d > 2$ and define the functions

$$\overline{H}_a(x; n) = \langle H_{B,\mathbb{Z}^d}(x; n)\rangle_\pi \qquad \text{and} \qquad \overline{H}_b(x) = \langle H_{*d,B}(x; 1)\rangle_\pi \tag{34}$$

where $n = [\frac{d-1}{2}]$ is the smallest positive integer for which $P^{(n)}_{\mathbb{Z}^d}(x)$ diverges as $x \to 0$. With the same calculation as in (22) we get

$$\frac{\overline{H}'_a(x)}{\overline{P}^{(n)}(x)} \leq 1 \qquad \text{and} \qquad \frac{\overline{H}'_b(x)}{P'_{*d}(x)} \leq 1. \tag{35}$$

We clearly have $(-1)^{n-1}\overline{H}_a(x) \leq (-1)^{n-1}P^{(n-1)}_{\mathbb{Z}^d}(x)$ and $\overline{H}_b(x) \leq \overline{P}(x)$ both with equality when $x = 0$. Since the functions $(-1)^{n-1}\overline{H}_a(x)$,$(-1)^{n-1}P^{(n-1)}_{\mathbb{Z}^d}(x)$, $\overline{H}_b(x)$ and $\overline{P}(x)$ are all decreasing functions of x we get with the same argument as in the proof of Lemma 2 that for a given $x \in]0, 1[$ there exists a $\xi \in]0, x[$ such that

$$1 \leq \frac{\overline{P}^{(n)}(\xi)}{P^{(n)}_{\mathbb{Z}^d}(\xi)} \qquad \text{and} \qquad 1 \leq \frac{P'_{*d}(\xi)}{\overline{P}'(\xi)}. \tag{36}$$

This extends Theorem 1 to random brushes and establishes the bounds (2).

5 Mean Field Theory

It is an obvious question to ask whether the full range of spectral dimensions allowed by (2) is realized for some random brushes. We do not have an answer to this question. However, in [3] the spectral dimensions for different classes of random combs were calculated exactly and shown to take the same values as in mean field theory [12]. By mean field theory we mean that the walk on the base (spine in the case of combs) always sees a new bristle drawn from the probability distribution μ whenever it is located at the root of a bristle. Since mean field theory is exact in one dimension we find it likely that it is also exact in higher dimensions where the walks are less likely to visit the same points on the base often. Mean field theory allows us to evaluate the spectral dimension very easily as we now explain.

The ensemble average of the function K_{G_1,G_2} defined in (19) can be written

$$\begin{aligned}\langle K_{B,\mathbb{Z}^d}(x;\omega)\rangle_\pi &= \Big\langle \prod_{t=1}^{|\omega|-1} \frac{2d}{2d+1-P_{F(\omega_t)}(x)} \Big\rangle_\pi \\ &\overset{\text{m.f.t.}}{=} \Big(\Big\langle \frac{2d}{2d+1-P_l(x)} \Big\rangle_\mu\Big)^{|\omega|-1}. \end{aligned} \tag{37}$$

where the second equality is the mean field theory approximation. The mean field theory approximation to the first return generating function is

$$\overline{P}_{\text{m.f.t.},d}(x) = \Big\langle \frac{2d}{2d+1-P_l(x)} \Big\rangle_\mu^{-1} P_{\mathbb{Z}^d}(x_{\text{ren}}(x)) \tag{38}$$

where $x_{\text{ren}}(x)$ is defined through

$$\sqrt{1-x_{\text{ren}}(x)} = \Big\langle \frac{2d}{2d+1-P_l(x)} \Big\rangle_\mu \sqrt{1-x}. \tag{39}$$

Now choose $\mu(l) = c_a l^{-a}$ with $a > 1$. The cases $d = 1$ and $d = 2$ we understand. Therefore consider the case $d \geq 3$. It is straightforward to calculate the asymptotic behaviour of the following derivatives:

$$\langle P_l^{(n)}(x)\rangle_\mu \sim x^{a/2-n} \qquad \text{for } n \geq 1, \tag{40}$$

$$x_{\text{ren}}(x) \sim \begin{cases} x^{a/2} \\ x \end{cases}, \qquad x'_{\text{ren}}(x) \sim \begin{cases} x^{a/2-1} & \text{if } 1 < a \leq 2 \\ 1 & \text{if } a > 2 \end{cases} \tag{41}$$

and

$$x_{\text{ren}}^{(n)}(x) \sim x^{a/2-n} \qquad \text{for } n \geq 2 \tag{42}$$

when $x \to 0$. We also see that the leading behaviour of the n-th derivative of (38) is

$$\overline{P}^{(n)}_{\text{m.f.t.},d}(x) \sim \langle P_l^{(n)}(x)\rangle_\mu + P^{(n)}_{\mathbb{Z}^d}(x_{\text{ren}}(x))(x'_{\text{ren}}(x))^n. \tag{43}$$

First consider the case $d = 3$, when we only have to look at the first derivative. Then $P'_{\mathbb{Z}^3}(x) \sim x^{-1/2}$ as $x \to 0$ and therefore

$$\overline{P}'_{\text{m.f.t.},3}(x) \sim \begin{cases} x^{a/4-1} & \text{if } 1 < a \leq 2 \\ x^{-1/2} & \text{if } a > 2 \end{cases} \tag{44}$$

which gives

$$d_s = \begin{cases} \frac{a}{2} + 2 & \text{if } 1 < a \leq 2 \\ 3 & \text{if } a > 2. \end{cases} \tag{45}$$

Doing the same for $d \geq 4$ we get the result

$$d_s = \begin{cases} a + 2 & \text{if } 1 < a \leq d - 2 \\ d & \text{if } a > d - 2. \end{cases} \tag{46}$$

It is easy to see that putting a single bristle on $\mathbb{Z}^d$ with probability distribution μ for $d \geq 4$ gives the same spectral dimension as mean field theory.

Now consider the random brush defined by $\mu(\infty) = p > 0$ and $\mu(0) = 1 - p$. It was shown in [3] that for $d = 1$ the spectral dimension of this random brush equals the spectral dimension of the full brush. The same is of course true for $d = 2$ and as well for $d \geq 4$, as was noted in the discussion below (31). Using mean field theory and similar analysis as above, we find that in any dimension the resulting random brush has also the same spectral dimension as the full brush. It is therefore clear that for this class of random brushes, if $d \neq 3$, mean field theory gives the correct spectral dimension. Settling the case $d = 3$ would require some extra work.

6 Conclusions

We have established bounds on the spectral dimensions of random graphs constructed by attaching linear graphs to $\mathbb{Z}^d$ and argued that mean field theory is likely to give the right value for the spectral dimension. The main monotonicity results are in fact valid for a much larger class of graphs as explained in Section 3; the base can be arbitrary and the bristles need only be recurrent graphs.

While our random brushes do contain loops, they are all on the base which is nonrandom and therefore do not yield much insight into how one might hope to bound or evaluate the spectral dimension of random graphs that contain loops like e.g. random surfaces. For such graphs we need to develop new techniques.

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