ON SITE PERCOLATION IN RANDOM QUADRANGULATIONS OF THE HALF-PLANE

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Abstract. We study site percolation on uniform quadrangulations of the upper half plane. The main contribution is a method for applying Angel's peeling process, in particular for analyzing an evolving boundary condition during the peeling. Our method lets us obtain rigorous and explicit upper and lower bounds on the percolation threshold p_c, and thus show in particular that $0.5511 \le p_c \le 0.5581$. The method can be extended to site percolation on other half-planar maps with the domain Markov property.

1. Introduction

Recent years have seen much progress on, and growing interest in, models of statistical physics defined on random maps embedded on surfaces. This includes work on percolation [1, 3, 19, 21], simple random walk [4, 9, 10, 11, 16] and also for example $O(n)$-loop models [6, 7] (see also the recent [13]). Angel's seminal paper [1] provided key tools for understanding both the local limits of planar maps themselves, as well as percolation on these local limits, by establishing a form of spatial Markovian property. This property is encapsulated in the so-called peeling process, which allows one to discover the map in a step-wise manner. The peeling process takes a particularly simple form for *half*-planar maps, where it has been used to analyze not only site-percolation on triangulations, but also for example edge- and face percolation on triangulations and quadrangulations [3]. The method has also been used for edge- and site-percolation on uniform planar maps as well as edge-percolation on uniform planar quadrangulations [19].

This paper applies the peeling process to analyze site percolation on uniform infinite half-planar quadrangulations (UIHPQ). Recall that site-percolation on a graph is defined by assigning one of two colours to each of the vertices, here called black and white, independently for different vertices. We denote the probability of colouring a vertex white by p and we assume this to be the same for all vertices. One is interested in whether or not the white connected cluster containing some particular site (in our case a fixed point 0 on the boundary) is almost surely finite, or if it is infinite with positive probability. Roughly speaking, one may use the peeling process to discover (part of) the map itself alongside the outer boundary of the percolation cluster. One keeps track of an 'active' part of the boundary of the cluster, whose size after n steps is denoted S_n, defined in such a way that the cluster is finite if and only if $S_n = 0$ for some n.

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In the well-understood case of site percolation on random half-planar *triangulations*, the process S_n is a random walk. Its i.i.d. increments have a distribution which can be found explicitly as a function of the percolation parameter p. By computing the drift of this random walk one arrives at the critical value p_c, which is $1/2$ for uniform triangulations (the value of p_c is known also for more general triangulations satisfying the domain Markov property [21]). Angel and Curien also obtained certain critical exponents by using this method [3].

A key aspect of the method is to identify an appropriate 'invariant boundary condition' which describes the colours of the vertices on the boundary on one particular side (usually the left) of the evolving percolation cluster. In the case of triangulations this boundary condition is deterministic (all black) and thus particularly simple. For edge-percolation on triangulations and quadrangulations, Angel and Curien used the invariance of the 'free' boundary condition. The fact that the invariant boundary condition is no longer deterministic makes the analysis slightly different, but S_n is still a random walk in this case.

As will be explained in much more detail below, the case of site percolation on quadrangulations presents further challenges. Not only is the invariant boundary condition not deterministic, it also exhibits complicated dependencies. In fact, we have not been able to describe it explicitly. The process S_n is no longer a Markov chain but needs to be analyzed as a function of the evolving boundary. In light of these difficulties it is not surprising that we obtain weaker results than in the models described above; our results regarding the percolation threshold p_c for site percolation on uniform quadrangulations are stated in Theorem 1.1. Apart from the results themselves, however, we hope that the methods we present in this work can be useful for future work on site percolation on other random maps.

1.1. **Problem setting and background.** We now recall the definition of the uniform infinite quadrangulation of the half-plane (UIHPQ). We start with an integer $m \geq 1$ and a $2m$-gon embedded in the sphere $\mathbb{S}^2$. We *root* this polygon by singling out an edge and an orientation of that edge, and we think of the face on the right of the root edge as the external face or 'outside'. Let $\phi(n, 2m)$ denote the number of quadrangulations of the inside of the polygon that have n internal vertices, viewed up to orientation-preserving homeomorphisms of the sphere. Thus $\phi(n, 2m)$ is a finite number. (By convention $\phi(0, 2) = 1$, counting the 'quadrangulation' consisting of one single edge only. Also, $\phi(n, m) = 0$ for odd m since quadrangulations are bipartite.) We define the uniform distribution $\theta_{n,2m}(\cdot)$ by assigning the same probability $1/\phi(n, 2m)$ to each such quadrangulation.

The UIHPQ is defined as the weak limit

$$\theta_{\infty,\infty} = \lim_{m\to\infty} \lim_{n\to\infty} \theta_{n,2m} \tag{1.1}$$

in the *local topology*. This topology is closely related to the one introduced by Benjamini and Schramm [10], and is defined by the following metric on embedded planar graphs. For any rooted graph M embedded in $\mathbb{S}^2$ and any $r \geq 1$, let $B_r(M)$ denote the embedded graph spanned by vertices at graph distance $\leq r$ from the root edge. For two such graphs M_1, M_2 define

$$d_{\mathrm{loc}}(M_1, M_2) = \frac{1}{1 + \sup\{r \geq 1 : B_r(M_1) = B_r(M_2)\}}, \tag{1.2}$$

where the equaility $B_r(M_1) = B_r(M_2)$ is interpreted in the sense of equivalence under deformations of the sphere, as before. The local topology is by definition the topology generated by the metric d_{loc}. Existence of the limits (1.1) in the local topology goes back to [1, 2] and [14]. We let $\mathbf{Q}$ denote a random variable sampled from the UIHPQ measure $\theta_{\infty,\infty}(\cdot)$ and note here that $\mathbf{Q}$ is almost surely an infinite rooted graph embedded in the plane, with an infinite simple boundary.

In analyzing the limit (1.1), the following combinatorial facts are of central importance [8]. First, the asymptotics of $\phi(n, 2m)$ are as follows:

$$\begin{aligned} \phi(n, 2m) &\sim C_{2m} \rho^n n^{-5/2}, \text{ as } n \to \infty, \\ C_{2m} &\sim K \alpha^{2m} m^{1/2}, \text{ as } m \to \infty, \end{aligned} \tag{1.3}$$

where $\rho = 12$, $\alpha = \sqrt{54}$ and K is a constant. The generating function $\sum_{n \geq 0} \phi(n, 2m) z^n$ is thus convergent for $|z| \leq 1/\rho$, and its value at $z = 1/\rho$ is known explicitly and denoted

$$Z(2m) = \sum_{n \geq 0} \phi(n, 2m) \rho^{-n} = \frac{8^m (3m-4)!}{(m-2)!(2m)!}. \tag{1.4}$$

(For $m = 1$ we interpret $Z(2)$ as its limiting value $4/3$.)

We now give a rough description of the peeling process, more details are given in Section 2. The boundary of $\mathbf{Q}$ may be identified with $\mathbb{Z}$ and each edge on the boundary with a pair $(i, i+1)$. By convention we take the root edge as $(-1, 0)$ pointing towards 0 (thus the quadrangulation is in the *upper* half plane). The peeling process proceeds by picking an edge on the boundary and 'discovering' the (unique) face f on its left. This face may have 0, 1 or 2 of its remaining 2 vertices on the boundary, see Figure 2. The probabilities of all the different possibilities for f may be computed explicitly using (1.3), and are given in terms of the numbers

$$\begin{aligned} q_{2k} = q_{2k+1} &= Z(2k+2)\rho^{-1}\alpha^{-2k}, \quad \text{for } k \geq 0, \\ q_{2k_1+1, 2k_2+1} &= \left(\frac{\rho}{\alpha}\right)^2 q_{2k_1} q_{2k_2}, \quad \text{for } k_1, k_2 \geq 0, \\ q_{-1} &= \left(\frac{\alpha}{\rho}\right)^2 = \frac{3}{8}. \end{aligned} \tag{1.5}$$

Roughly speaking, q_{-1} is the probability that f has no further vertices on the boundary, q_k is the probability that it has one further vertex on the boundary at distance k from the peeling edge, and q_{k_1,k_2} is the probability that it has two further vertices on the boundary at distances k_1 and k_2, see (2.3) and (2.4). The face f divides $\mathbf{Q}$ into two parts, one 'above' f and one 'below'. We may redefine the boundary by 'forgetting' the lower part. It is a consequence of the domain Markov property that the remaining, upper part also has the law of $\mathbf{Q}$. One may thus continue to discover the rest of $\mathbf{Q}$ by repeating the steps above.

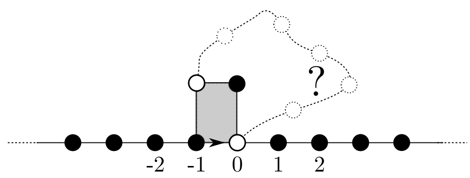

FIGURE 1. A new white vertex is discovered by exposing the gray face. At this stage it is not known whether it belongs to the white cluster containing 0. A possible existence of such a connection is indicated by dashed lines.

Similarly to the approach pioneered by Angel [1], we will couple the peeling process with the discovery of the boundary of the percolation cluster. Before starting the peeling, we begin by colouring all vertices on the boundary $\mathbb{Z}$ black, except for the vertex 0 which is coloured white. Each time we discover a face we will sample (independently of everything else) the colours black/white of any vertices on f not discovered in a previous step. We describe this procedure fully in Section 2.1, but as an example of what may happen, suppose that at the very first peeling move we choose the root edge $(-1, 0)$ to peel from, and that the face f discovered has two of its vertices in the interior of $\mathbf{Q}$, see Fig. 1. Suppose, furthermore, that these vertices receive colours black and white, counting counterclockwise starting from 0. At this stage we cannot determine whether or not the new white vertex will be part of the white cluster containing 0 or not. (This differs markedly from the case of triangulations, where the face f could have at most one new vertex and this new vertex would have had to be part of the cluster if it was white.) We will deal with this by storing, at each step of the peeling process, information about the 'yet-to-be-decided' white vertices in what we call a *mixed boundary*. At step n we denote the mixed boundary by $X_n = (X_n(j) : j \geq 1)$, which is a sequence of colours black and white, indexed by $\mathbb{N}$. Understanding the process $(X_n)_{n\geq 0}$ is central to our approach.

1.2. **Main results.** For each sample of $\mathbf{Q}$ and of the colours black/white, let C_0 denote the connected component in the white subgraph containing the boundary point 0, and let $|C_0|$ denote its size (number of vertices, say). It turns out (as we will show) that there is a number $p_{\mathrm{c}} \in [0, 1]$ such that $\mathbb{P}(|C_0| = \infty) = 0$ if $p < p_{\mathrm{c}}$ and $\mathbb{P}(|C_0| = \infty) > 0$ if $p > p_{\mathrm{c}}$. Our primary objective has been to pinpoint the value of the percolation threshold p_{c}. (It is is clear that for each $\mathbf{Q}$, the conditional probability that $|C_0| = \infty$ given $\mathbf{Q}$ is weakly increasing in the percolation parameter p. Thus there is a critical value $p_{\mathrm{c}}(\mathbf{Q}) \in [0, 1]$ such that this probability is positive if $p > p_{\mathrm{c}}(\mathbf{Q})$ and zero if $p < p_{\mathrm{c}}(\mathbf{Q})$. Yet the existence of p_{c} as defined above requires an argument since there is no obvious monotonicity when we average also over $\mathbf{Q}$.)

In our attempts to determine the value of p_{c}, we were led to analyze the mixed boundary $(X_n)_{n\geq 0}$. As we will see, this is a Markov process, and it has a stationary limiting distribution. We denote a sample from this limiting distribution by $\xi^{(p)} = (\xi^{(p)}(j) : j \geq 1)$. Also define for all $k \geq 1$

$$q'_{2k} = q_{2k} + \sum_{\substack{k_1+k_2=2k \\ k_1,k_2 \geq 1,\text{odd}}} q_{k_1,k_2}. \tag{1.6}$$

In what follows, white vertices are denoted $\circ$ and black vertices $\bullet$. The following are the results we have obtained.

Theorem 1.1. *The percolation threshold* p_{c} *satisfies*

$$p_{\mathrm{c}} \begin{cases} \geq \sup\{p \in [0,1] : \alpha(p) < 0\} \\ \leq \inf\{p \in [0,1] : \alpha(p) > 0\}, \end{cases} \tag{1.7}$$

where

$$\alpha(p) = \frac{3}{8}p^2 + \frac{5}{8}p - \frac{1}{2} + \beta(p) \tag{1.8}$$

and

$$\beta(p) = \Big(p + \frac{1}{3}\Big) \sum_{k \geq 1,\text{odd}} \mathbb{P}(\xi^{(p)}(k+1) = \circ) q_k + \sum_{k \geq 1,\text{even}} \mathbb{P}(\xi^{(p)}(k+1) = \circ) q'_k. \tag{1.9}$$

The function $\alpha(p)$ depends on the distribution of $\xi^{(p)}$, which we have not been able to find explicitly. However we do derive upper and lower bounds on the probabilities $\mathbb{P}(\xi^{(p)}(k+1) = \circ)$, resulting in bounds on p_{c}. In particular, we obtain the following:

Proposition 1.2. *It holds that* $0.511 \approx \frac{\sqrt{493}-13}{18} \leq p_{\mathrm{c}} \leq \frac{\sqrt{73}-5}{6} \approx 0.591$.

In fact, we present a method for deriving better bounds for p_{c}, which when implemented numerically on a computer gives the bounds

$$0.5511 \leq p_{\mathrm{c}} \leq 0.5581 \tag{1.10}$$

mentioned in the Abstract. (The bounds presented in Proposition 1.2 were chosen as they are simple algebraic expressions obtained without recourse to numerical methods.) Our basic approach is to find probability measures which are stochastically above and below the law of $\xi^{(p)}$, thus giving upper and lower bounds on $\alpha(p)$. These measures are obtained as the stationary distributions of certain finite state space Markov chains, meaning that they can be found by solving deterministic equations involving p. We expect that the gap in (1.10) could be narrowed by further increasing the size of the state space. Note that we do not necessarily expect p_{c} to be given by a simple formula, indeed our bounds quickly become too complicated to write down by hand, which is why we have used numerical methods.

It also follows from our analysis that the value p_{c} is the same for a variety of boundary conditions other than all-black, such as the 'free' boundary condition (see Proposition 2.3).

1.3. **Outline.** In Section 2 we properly define the peeling process, and also provide some basic results about the critical probability p_{c}. In Section 3 we study the process $(X_n)_{n\geq 0}$ in detail. We then apply our results on $(X_n)_{n\geq 0}$ in Section 4, where we prove Theorem 1.1 and Proposition 1.2. Furthermore, we explain how one may obtain increasingly better upper and lower bounds on p_c.

We note here that often the dependency on p will be dropped in the notation as e.g. in X_n and S_n.

2. Peeling process

The peeling process gives a sequence $\mathbf{Q} = \mathbf{Q}_0 \supset \mathbf{Q}_1 \supset \cdots$ of random infinite quadrangulations with infinite simple boundary. At each step $n \geq 0$ there is a choice of an edge r_n on the boundary of $\mathbf{Q}_n$, which we require to be independent of $\mathbf{Q}_n$ itself, and given this edge we obtain $\mathbf{Q}_{n+1}$ by a random operation Peel such that

$$\mathbf{Q}_{n+1} = \mathrm{Peel}(\mathbf{Q}_n, r_n). \tag{2.1}$$

The operation Peel is defined as follows. We start by discovering the unique face f_n of $\mathbf{Q}_n$ adjacent to r_n (the distribution of f_n will be given shortly). This face may be adjacent to 0, 1 or 2 internal vertices, the remaining vertices are on the boundary of $\mathbf{Q}_n$ and may be to the left or to the right of r_n, as illustrated in Figure 2. If f_n has vertices on the boundary then it encloses one or two subquandrangulations of $\mathbf{Q}_n$, each with a finite, simple boundary. By definition, $\mathbf{Q}_{n+1}$ is the infinite quadrangulation obtained by removing f_n along with any such enclosed subquadrangulations. The edges and vertices of f_n that then become part of the boundary of $\mathbf{Q}_{n+1}$ are called *exposed* edges and vertices, respectively. The number of exposed edges is denoted by $\mathcal{E}_n$. The edges on the boundary of $\mathbf{Q}_n$ which are enclosed by f_n are called *swallowed edges*, and the numbers of such edges to the left and right of r_n are denoted by $\mathcal{R}_n^-$ and $\mathcal{R}_n^+$, respectively.

Recall the q:s in (1.5) and (1.6). We note for future reference that they satisfy

$$\begin{aligned}
&\sum_{k\geq 0} q_{2k+1} = \sum_{k\geq 0} q_{2k} = \tfrac{1}{8}, \qquad \sum_{k_1,k_2\geq 0} q_{2k_1+1,2k_2+1} = \tfrac{1}{24} \\
&\sum_{k\geq 1} q'_{2k} = \tfrac{1}{8} - \tfrac{1}{9} + \tfrac{1}{24} = \tfrac{1}{18}.
\end{aligned} \tag{2.2}$$

As long as the peeling edge r_n is always chosen independently of $\mathbf{Q}_n$ we have the following [3]:

- $\mathbf{Q}_n$ is distributed as $\mathbf{Q}$ for all $n \geq 0$ and is independent of the previous steps.
- The couples $(\mathcal{E}_n, \mathcal{R}_n^-, \mathcal{R}_n^+)$ form an i.i.d. sequence, each being independent of $\mathbf{Q}_n$.
- If the revealed face f_n contains vertices in the boundary on both sides of r_n then $\mathcal{R}_n^+ > 0$ and $\mathcal{R}_n^- > 0$ and then necessarily $\mathcal{E}_n = 1$. We have
$$\mathbb{P}(\mathcal{E}_n = 1, \mathcal{R}_n^+ = k_1, \mathcal{R}_n^- = k_2) = q_{k_1,k_2} \tag{2.3}$$
for $k_1, k_2 \geq 1$ both odd.
- Otherwise either $\mathcal{R}_n^- = 0$ or $\mathcal{R}_n^+ = 0$ and then we have

$$\mathbb{P}(\mathcal{E}_n = e, \mathcal{R}_n^{\pm} = k) = \begin{cases} q_{-1} = \frac{3}{8}, & e = 3, k = 0, \\ q_k \mathbf{1}_{\{k \text{ odd}\}}, & e = 2, k \geq 1, \\ q'_k \mathbf{1}_{\{k \text{ even}\}} + \frac{1}{3} q_k \mathbf{1}_{\{k \text{ odd}\}}, & e = 1, k \geq 1. \end{cases} \tag{2.4}$$

It follows from this and (2.2) that

$$\mathbb{P}(\mathcal{E}_n = 2, \mathcal{R}_n^{\pm} = 0) = \tfrac{1}{8}, \quad \mathbb{P}(\mathcal{E}_n = 1, \mathcal{R}_n^{\pm} = 0) = \tfrac{5}{18}. \tag{2.5}$$

- The expectations $\mathbb{E}(\mathcal{E}_n) = 2$ and $\mathbb{E}(\mathcal{R}_n^{\pm}) = 1/2$.

The enclosed subquadrangulations which are removed to form $\mathbf{Q}_{n+1}$ from $\mathbf{Q}_n$ are almost surely finite, and independent both of each other (if there are two) and of $\mathbf{Q}_{n+1}$. If the perimiter of an enclosed quadrangulation is $2m$, then (1.4) gives the partition function of its distribution, but in this paper we shall only use the fact that it is almost surely finite.

2.1. **Percolation and peeling.** Recall that the boundary of $\mathbf{Q}$ is identified with $\mathbb{Z}$, and that the directed edge from -1 to 0 is taken as root. We will consider percolation on $\mathbf{Q}$ with various boundary conditions, i.e. different ways of assigning black and white colours to the vertices on the boundary $\mathbb{Z}$. The vertex 0 is always coloured white, and the vertices $1, 2, 3, \dots$ on the boundary to the right of 0 are always coloured black. The vertices $-1, -2, -3, \dots$ on the boundary to the left of 0 receive some random or deterministic colours, described by a vector $\xi \in \Sigma := \{\bullet, \circ\}^{\{1,2,3,\dots\}}$. Here $\xi(j) \in \{\bullet, \circ\}$ denotes the colour of $-j$.

For reasons that will appear later we mainly consider ξ which are *admissible*, which we define to mean that (i) $\xi(1) = \bullet$, and (ii) for all $k \geq 1$, if $\xi(k) = \circ$ then $\xi(k+1) = \bullet$ (i.e., there are no adjacent white vertices). We let $\hat{\Sigma} \subseteq \Sigma$ denote the set of all admissible boundary conditions. The canonical admissible boundary condition is obtained when all vertices to the left of 0 are black, as described in Section 1; we sometimes write this as $\xi \equiv \bullet$.

The remaining vertices, i.e. those not on the boundary, are coloured independently of each other and of ξ, each being white with probability p and black with probability $1 - p$.

We are interested in knowing for which values of p the white cluster C_0 percolates, i.e. is infinite with positive probability. We will investigate this using the peeling process where we discover the colours at the same time as we peel. It remains to define the peeling edges r_n. In the first step we let $r_0 = (-1, 0)$. Assuming that we have defined $r_n = (r_n^L, r_n^R)$ for some $n \geq 0$, we reveal the new face f_n and then reveal the colours of all exposed vertices on f_n. There will be a few different cases for the next peeling edge r_{n+1}. The first case we consider is when $\mathcal{R}_n^+ = 0$, that is to say that no edges on the boundary to the right of r_n are swallowed. Starting from the vertex immediately to the left of r_n^R in the boundary of $\mathbf{Q}_{n+1}$, follow this boundary from right to left until the first black vertex is encountered. This black vertex is denoted r_{n+1}^L, the vertex immediately to its right in the boundary of $\mathbf{Q}_{n+1}$ is denoted r_{n+1}^R, and $r_{n+1} = (r_{n+1}^L, r_{n+1}^R)$ is taken as the new peeling edge. If, on the other hand, $\mathcal{R}_n^+ > 0$, then we denote by s_n the rightmost vertex on the boundary of $\mathbf{Q}_n$ belonging to the new face f_n. We then follow the boundary of $\mathbf{Q}_{n+1}$ from right to left starting at the vertex immediately to the left of s_n, until we discover a black vertex which we take to be r_{n+1}^L. The vertex immediately to its right is taken to be r_{n+1}^R. Examples are given in Figure 2. Note that the choice of r_{n+1} is always independent of $\mathbf{Q}_{n+1}$.

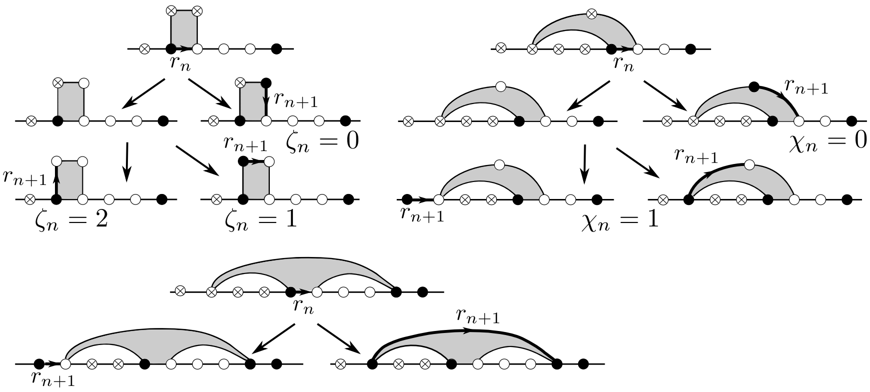

FIGURE 2. Examples of the definition of the peeling edge r_n, as well as the random variables ζ_n and χ_n (defined in Section 4.1). Unspecified colours are represented by $\otimes$.

In each step, $\mathbf{Q}_n$ will have a *mixed–white–black* boundary condition. The *white* part of the boundary is possibly empty; if non-empty it leftmost endpoint is marked by r_n^R. We will denote the number of white vertices in this white arc *which belong to* C_0 by S_n. Thus $S_0 = 1$ and if $S_n = 0$ then $S_{n'} = 0$ for all $n' \geq n$. The mixed part of the boundary is described by a vector $X_n = (X_n(j) : j \geq 1) \in \Sigma$, where $X_n(j)$ denotes the colour of the j:th vertex to the left of r_n^R. In fact it is easy to see that if $\xi = X_0$ is admissible then X_n is admissible for all $n \geq 0$.

Although we have defined this procedure for all $n \geq 0$, we are primarily concerned with the process up to the first time that $S_n = 0$ (if this ever happens). Until this time, all vertices in the *white* arc contribute to S_n, and the state of the mixed boundary in $\mathbf{Q}_n$ is thus determined by $(X_n, S_n) \in \hat{\Sigma} \times \mathbb{N}$. This is easily seen to be a Markov chain. Note that the process $(X_n)_{n\geq 0}$ is itself a Markov chain, but the process $(S_n)_{n\geq 0}$ is not; whereas X_{n+1} only depends on X_n as well as the next peeling move, to determine S_{n+1} we must also look at X_n since some of the white vertices in X_n may become part of the white cluster.

The key observation, which this model shares with other studies of percolation using the peeling process, is that C_0 must be finite if ever $S_n = 0$. Note that this holds even if the initial boundary condition ξ contains infinitely many white vertices, since only finitely many of them can become connected to 0 during a finite number of peeling steps. We omit formal proof of this but state it as a proposition:

Proposition 2.1. *If* $S_n = 0$ *for some* n *then* C_0 *is finite.*

2.2. **Percolation threshold for different boundary conditions.** Recall that the state of the left side of the boundary at time $n = 0$ is given by a random or deterministic element ξ of $\hat{\Sigma}$. We will denote the distribution of ξ by ν.

In this section we show that there is indeed a percolation threshold $p_{\mathrm{c}}^{\bullet}$ when the boundary condition $\xi \equiv \bullet$ is all-black (as stated in Section 1). We will also show that p_{c} is in fact the same for all distributions ν in a certain class which we call *amenable*.

The following result is not new but we include a proof for completeness.

Proposition 2.2. *For each ν supported on $\hat{\Sigma}$ we have that*

$$\text{if } p \text{ is such that } \mathbb{P}(S_n \to 0) = 1 \text{ then } \mathbb{P}(|C_0| = \infty) = 0 \text{ for all } p' \le p, \tag{2.6}$$

and

$$\text{if } p \text{ is such that } \mathbb{P}(S_n \to 0) < 1 \text{ then } \mathbb{P}(|C_0| = \infty) > 0 \text{ for all } p' \ge p. \tag{2.7}$$

In particular, there is a value $p_c^\nu \in [0,1]$ such that $\mathbb{P}(|C_0| = \infty) = 0$ if $p < p_c^\nu$, and $\mathbb{P}(|C_0| = \infty) > 0$ if $p > p_c^\nu$.

Here p_c^ν may be identified either as the supremum of p's as in (2.6), or the infimum of p's as in (2.7).

Proof. Write $P_p^{\mathbf{Q}}$ for the conditional measure $\mathbb{P}(\cdot \mid \mathbf{Q})$, thus $P_p^{\mathbf{Q}}$ governs the black/white colours only. An obvious coupling gives, for each $\mathbf{Q}$, that

$$\text{if } p' \le p \text{ then } P_{p'}^{\mathbf{Q}}(|C_0| = \infty) \le P_p^{\mathbf{Q}}(|C_0| = \infty). \tag{2.8}$$

To see (2.6), suppose p is such that $\mathbb{P}(S_n \to 0) = 1$. By Proposition 2.1 we thus have that

$$0 = \mathbb{P}(|C_0| = \infty) = \mathbb{E}[P_p^{\mathbf{Q}}(|C_0| = \infty)]. \tag{2.9}$$

Hence $P_p^{\mathbf{Q}}(|C_0| = \infty) = 0$ almost surely, and so by (2.8) also $P_{p'}^{\mathbf{Q}}(|C_0| = \infty) = 0$ almost surely, which gives the result.

For (2.7), first let A be the event that, during the peeling process, there are infinitely many times when the revealed face f_n has two exposed vertices and they are both white. Since the sequence of revealed faces and the colours of the exposed vertices is i.i.d. it follows that $\mathbb{P}(A) = 1$. It follows that for p as in (2.7) we have

$$\mathbb{P}(|C_0| = \infty) \ge \mathbb{P}(\{S_n > 0 \; \forall n \ge 0\} \cap A) > 0. \tag{2.10}$$

This means that $\mathbb{E}[P_p^{\mathbf{Q}}(|C_0| = \infty)] > 0$ and hence (using (2.8)) for all $p' \ge p$ also $\mathbb{E}[P_{p'}^{\mathbf{Q}}(|C_0| = \infty)] > 0$, as required. $\square$

Next we recall the concept of stochastic ordering. We order the elements of Σ by saying that $\xi \le \xi'$ if whenever $\xi(j) = \circ$ then also $\xi'(j) = \circ$. We say that an event (subset) $A \subseteq \Sigma$ is *increasing* if $\xi \in A$ implies that $\xi' \in A$ whenever $\xi \le \xi'$. For two probability measures μ, μ' on Σ we say that μ' (stochastically) *dominates* μ if for all increasing events A we have that $\mu(A) \le \mu'(A)$. We will denote the probability measure which assigns the values $\xi(j) \in \{\bullet, \circ\}$ independently, with probability p for $\circ$, by $\textsc{iid}(p)$. (Note that ξ is a.s. not admissible under $\textsc{iid}(p)$ unless $p = 0$.)

We say that the random, admissible boundary condition $\xi \in \hat{\Sigma}$, or equivalently its distribution ν, is *amenable* if there exists $p' < 1$ such that ν is dominated by $\textsc{iid}(p')$. Note that $\xi \equiv \bullet$ is amenable for any $p' < 1$.

Proposition 2.3. *If ν is amenable then $p_c^\nu = p_c^\bullet$.*

We will write simply p_c for the common critical value.

Proof. In this proof we let $\mathbb{P}$ denote the probability measure under which ξ has law ν. We need to show that, firstly, if $p > p_c^\bullet$ then $\mathbb{P}(|C_0| = \infty) > 0$, and secondly, if $p < p_c^\bullet$ then $\mathbb{P}(|C_0| = \infty) = 0$.

The first statement is clear: for each joint realization of $\mathbf{Q}$ and the black/white colours such that $|C_0| = \infty$ when the boundary is all-black, then also $|C_0| = \infty$ for the boundary condition ξ. We now turn to the second statement.

Let $\mathbf{Q}^\boxtimes$ denote the *matching lattice* of $\mathbf{Q}$, that is the graph obtained from $\mathbf{Q}$ by adding to the edge set both diagonals of each face (this graph is in general not planar). Then C_0 is finite if and only if there are vertices $-i < 0$ and $j > 0$ on the boundary $\mathbb{Z}$ such that both are black (equivalently, $\xi(i) = \bullet$) and there is a path from $-i$ to j in $\mathbf{Q}^\boxtimes$ traversing only black vertices, see e.g. [18, Section 2.2] or [15, Section 11.10]. We call such a path a *blocking circuit*. We need to show that with probability 1 there is a blocking circuit when the boundary condition is ξ.

For each $j \in \mathbb{Z}$, let B_j denote the event that there are $i < j$ and $k > j$ such that there is a path in $\mathbf{Q}^\boxtimes$ from i to k which (i) contains no other vertex of $\mathbb{Z}$, and (ii) contains only black vertices, apart from possibly i and k. Since $p < p_c^\bullet$ we have that $\mathbb{P}(B_0) = 1$. By invariance under translation of $\mathbf{Q}$ with respect to the boundary we thus have that $\mathbb{P}(B_j) = 1$ for all $j \in \mathbb{Z}$. Consider the event $B = \cap_{j<0} B_j$. We have that $\mathbb{P}(B) = 1$, and on the event B there are infinitely many $i > 0$ such that there is a $\mathbf{Q}^\boxtimes$-path from $-i$ to some $j > 0$ which contains no other vertex of $\mathbb{Z}$ and consists only of black vertices, apart from possibly $-i$. (To see this, note that some of the paths whose existence are guaranteed by the B_j may 'merge'.) Since ξ is amenable, at least one (in fact infinitely many) of these vertices $-i$ are black. Hence there is a blocking circuit with probability one. $\square$

The argument for Proposition 2.3 applies more generally, e.g. it is enough if for any fixed sequence $i_1 < i_2 < i_3 < \cdots$, with ν-probability 1 at least one $\xi(i_k) = \bullet$. One can also adapt the argument to cases when one has a different boundary condition than all-black to the right of 0, e.g. the 'free' boundary condition when both sides are indepenently $\textsc{iid}(p)$. Similar results on equality of percolation thresholds under different boundary conditions have been obtained in e.g. [3, 21]. The arguments there are different and are formulated only for all-black and $\textsc{iid}(p)$ boundary conditions.

3. Evolution of the mixed boundary

Let $\Sigma^* = \bigcup_{k\geq 1}\{\bullet, \circ\}^k$ denote the set of all *finite* sequences of $\bullet$'s and $\circ$'s of length at least 1, and let $\hat{\Sigma}^*$ denote the subset of Σ^* consisting of all sequences such that the first bit is $\bullet$ and such that there are no adjacent $\circ$'s. Clearly Σ^* and $\hat{\Sigma}^*$ are countable. We endow Σ with the product topology and $\hat{\Sigma}$ with the subspace topology. Note that we may define a metric d_Σ on Σ which generates its topology by e.g.

$$d_\Sigma(x, y) = (\sup\{k \ : \ [x]_k = [y]_k\} + 1)^{-1} \tag{3.1}$$

where $[x]_k \in \Sigma^*$ denotes the vector consisting of the first k entries in x. Since $\hat{\Sigma}$ is closed in Σ and Σ is compact, it follows that also $\hat{\Sigma}$ is compact.

We now turn to investigating existence and uniqueness of invariant distributions for the process $(X_n)_{n\geq 0}$. We begin by noting the following immediate consequence of the Stone–Weierstrass theorem:

Lemma 3.1. *Let $C(\hat{\Sigma},\mathbb{R})$ be the set of continuous real valued functions on $\hat{\Sigma}$ equipped with the uniform topology. Let $C^*(\hat{\Sigma},\mathbb{R})$ be the subalgebra of functions which only depend on finitely many coordinates. Then $C^*(\hat{\Sigma},\mathbb{R})$ is dense in $C(\hat{\Sigma},\mathbb{R})$.*

Next, recall that a Markov process $(Y_n)_{n\geq 0}$ on $\hat{\Sigma}$ is *Feller* if for any $n \geq 0$ and any bounded continuous function $g : \hat{\Sigma} \to \mathbb{R}$, the function $\xi \mapsto \mathbb{E}_\xi(g(Y_n))$ is continuous in ξ (where $\mathbb{E}_\xi$ is the expected value given that $Y_0 = \xi$).

Lemma 3.2. *The process $(X_n)_{n\geq 0}$ is Feller.*

Proof. Fix $\xi \in \hat{\Sigma}$, $n \geq 0$ and a bounded continuous $g : \hat{\Sigma} \to \mathbb{R}$. Let $\epsilon > 0$. We will show that there exists $\delta > 0$ such that

$$|\mathbb{E}_\xi(g(X_n)) - \mathbb{E}_{\xi'}(g(X_n))| < \epsilon \tag{3.2}$$

for every $\xi' \in \hat{\Sigma}$ such that $d_\Sigma(\xi,\xi') < \delta$. Let

$$A_n(k) = \left\{\sum_{i=0}^{n} \mathcal{R}_i^- \leq k\right\} \tag{3.3}$$

i.e. $A_n(k)$ is the event that we swallow no more than k edges on the left in the first $n+1$ steps. Using that g is bounded, let $C > 0$ be a constant such that $\sup g < C$. Since the $\mathcal{R}_i^-$ are a.s. finite we may choose k large enough such that $\mathbb{P}(A_n^c(k)) < \epsilon/(4C)$. Since g is continuous, by Lemma 3.1 one may choose j large enough such that there is a function $g_j \in C^*(\Sigma,\mathbb{R})$ which depends only on the first j coordinates obeying $\sup|g - g_j| < \epsilon/4$. Then

$$\begin{aligned}
|\mathbb{E}_\xi(g(X_n)) - \mathbb{E}_{\xi'}(g(X_n))| &\leq |\mathbb{E}_\xi(g(X_n))\mathbf{1}_{\{A_n(k)\}}) - \mathbb{E}_{\xi'}(g(X_n)\mathbf{1}_{\{A_n(k)\}})| \\
&\quad + |\mathbb{E}_\xi(g(X_n)\mathbf{1}_{\{A_n^c(k)\}}) - \mathbb{E}_{\xi'}(g(X_n)\mathbf{1}_{\{A_n^c(k)\}})| \\
&\leq |\mathbb{E}_\xi(g_j(X_n)\mathbf{1}_{\{A_n(k)\}}) - \mathbb{E}_{\xi'}(g_j(X_n)\mathbf{1}_{\{A_n(k)\}})| \\
&\quad + 2\sup|g - g_j| + 2C\mathbb{P}(A_n^c(k)) \\
&< |\mathbb{E}_\xi(g_j(X_n)\mathbf{1}_{\{A_n(k)\}}) - \mathbb{E}_{\xi'}(g_j(X_n)\mathbf{1}_{\{A_n(k)\}})| \\
&\quad + \epsilon.
\end{aligned} \tag{3.4}$$

Now choose $\delta < 1/(1+j+k)$. Then, if $d_\Sigma(\xi,\xi') < \delta$ it holds that $[\xi]_{j+k} = [\xi']_{j+k}$. Since g_j depends only on the first j coordinates it thus holds that $\mathbb{E}_\xi(g_j(X_n)\mathbf{1}_{\{A_n(k)\}}) = \mathbb{E}_{\xi'}(g_j(X_n)\mathbf{1}_{\{A_n(k)\}})$ and thus

$$|\mathbb{E}_\xi(g(X_n)) - \mathbb{E}_{\xi'}(g(X_n))| < \epsilon. \tag{3.5}$$

□

Proposition 3.3. *There is a probability measure μ on $\hat{\Sigma}$ which is invariant for the process $(X_n)_{n\geq 0}$.*

Proof. This follows from the Krylov–Bogolyubov Theorem [17, Corollary 4.18] since $\hat{\Sigma}$ is compact and $(X_n)_{n\geq 0}$ is Feller. □

Do we expect the invariant distribution μ to be unique? Imagine placing at time 0 a 'flag' on the edge immediately to the left of -1. At time n the flag will have moved away from the peeling edge or been swallowed. Each time it is swallowed we reset it immediately to the left of -1. If the position of the flag is 'recurrent enough' then X_n will always retain information about the initial condition ξ, and hence in this case the distribution μ cannot be unique. We make this intuitive sketch more precise now.

As mentioned, at time $n = 0$ we mark the edge $(-2, -1)$ just left of r_0 by a 'flag'. During the peeling, the relative position of the flagged edge with respect to the peeling edge r_n will change: the distance increases when we 'input' into X_n, and decreases when we swallow to the left. Whenever we swallow the flagged edge we reset the flag on the edge just to the left of the peeling edge r_n. We denote the distance between r_n and the flagged edge by W_n. Thus $W_0 = 1$ and $W_n \geq 1$ for all $n \geq 0$. Let $X_n^* \in \hat{\Sigma}^*$ denote the vector of colours black/white of the vertices between r_n and the flagged edge. So the length $|X_n^*|$ of X_n^* is precisely W_n.

Note that the process $(X_n^*)_{n\geq 0}$ does not depend on the initial state $\xi \in \hat{\Sigma}$ of $(X_n)_{n\geq 0}$. In fact, the evolution of X_n^* does not depend on $X_n(k)$ for any $k > W_n$. Thus $(X_n^*)_{n\geq 0}$ is a Markov chain on the countable state space $\hat{\Sigma}^*$, and it is not hard to see that it is irreducible and aperiodic. Hence it is either transient, null-recurrent, or positive-recurrent, depending on p. By a slight abuse of terminology, we will refer to these three cases as the *transient*, *null-recurrent*, and *positive-recurrent* cases, respectively, also when referring to the chain $(X_n)_{n\geq 0}$ itself.

For $\xi^* \in \hat{\Sigma}^*$, with length $\ell = |\xi^*|$, and $\xi \in \hat{\Sigma}$, define the *concatenation* of ξ^* and ξ as the element $\bar{\xi}$ of Σ given by

$$\bar{\xi}(j) = \begin{cases} \xi^*(j), & \text{if } j \leq \ell, \\ \xi(j-\ell), & \text{if } j > \ell. \end{cases}$$

Consider now the positive-recurrent case. Then standard Markov chain theory [20] implies that $(X_n^*)_{n\geq 0}$ has a unique invariant distribution μ^* supported on $\hat{\Sigma}^*$. Let $\xi^* \in \hat{\Sigma}^*$ denote a random variable with distribution μ^*. One may obtain many distributions μ which are invariant for the 'whole' process $(X_n)_{n\geq 0}$ by concatenating ξ^* with some $\xi \in \hat{\Sigma}$. Let $\mu^{(p)}$ be the probability measure on $\hat{\Sigma}$ which is given as the distribution of ξ^* concatenated with $\xi \equiv \bullet$. Then $\mu^{(p)}$ is clearly an invariant measure for $(X_n)_{n\geq 0}$. Equivalently, $\mu^{(p)}$ is the limiting distribution of the process $(X_n)_{n\geq 0}$ starting with $X_0 \equiv \bullet$.

We now turn to the transient and null-recurrent cases. For any probability measure ν on $\hat{\Sigma}$, let $[\nu]_k$ denote the law of $[\xi]_k \in \{\bullet, \circ\}^k$ when ξ has distribution ν. We define the total-variation distance between two probability measures μ and ν on $\hat{\Sigma}$ by

$$\|\mu - \nu\|_{\mathrm{TV}} = \sup_A |\mu(A) - \nu(A)| \tag{3.6}$$

where the supremum is taken over all measurable sets A.

Proposition 3.4. *Let $\xi \in \hat{\Sigma}$ be random or deterministic and let $(X_n)_{n\geq 0}$ denote the chain started in ξ. Let μ be an invariant distribution as indentified in Proposition 3.3. In the transient and null-recurrent cases, we have for each $k \geq 1$ that*

$$\|[X_n]_k - [\mu]_k\|_{\mathrm{TV}} \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

In particular, μ is unique and may be obtained as the limiting distribution starting in any state.

In these cases we denote the unique invariant distribution by $\mu^{(p)}$. Thus $\mu^{(p)}$ is uniquely defined for all $p \in [0,1]$.

Proof. We may couple the chain $X_n^{(\mu)}$ started in μ with the chain $X_n^{(\xi)}$ started in ξ in the natural way, by using the same peeling moves. As noted above, the flag distances are the same in both processes, we denote this common value by W_n. Also, for all n we have that $X_n^{(\mu)}(k) = X_n^{(\xi)}(k)$ for all $k \leq W_n$. It follows from the coupling inequality that

$$\|[X_n^{(\xi)}]_k - [\mu]_k\|_{\mathrm{TV}} \leq \mathbb{P}([X_n^{(\xi)}]_k \neq [X_n^{(\mu)}]_k) \leq \mathbb{P}(W_n < k). \tag{3.8}$$

The event $\{W_n < k\}$ is precisely the same as the event that X_n^* belongs to the finite subset $\{\xi \in \hat{\Sigma}^* : |\xi| < k\}$ of $\hat{\Sigma}^*$. By standard results for Markov chains, e.g. [20, Theorem 1.8.5], it follows that $\mathbb{P}(W_n < k) \to 0$ as $n \to \infty$ in the transient and null-recurrent cases, proving (3.7).

Using Prohorov's theorem [12, Section I.5] (or Lemma 3.1 straight away) we deduce from (3.7) that $X_n^{(\xi)}$ converges weakly to μ for all choices of ξ, thereby proving also the final part of the statement. $\square$

By Proposition 3.4 (and by definition, in the positive-recurrent case) for all p the measure $\mu^{(p)}$ can be obtained as the limiting distribution when starting with initial condition $\xi \equiv \bullet$. We let $\xi^{(p)} \in \hat{\Sigma}$ denote a random variable with distribution $\mu^{(p)}$. The next result implies that the stationary boundary condition $\xi^{(p)}$ is amenable whenever $p < 1$:

Lemma 3.5. *For all p, the measure $\mu^{(p)}$ is dominated by* IID*(p).*

Proof. It suffices to show that if $X_0 \equiv \bullet$ then the law of X_n is dominated by IID(p) for all $n \geq 0$. More precisely, we will show that one may define a process $(Y_n)_{n\geq 0}$ such that for all n, (a) the distribution of Y_n is IID(p) and (b) $X_n \leq Y_n$. We show this by induction.

For $n = 0$ this clearly holds if we just sample Y_0 from IID(p). Assume that we have such a coupling for the first n steps in the peeling process. There are three main cases to consider depending on the next peeling move. In the first case, $X_{n+1} = X_n$ and we may take also $Y_{n+1} = Y_n$ (this happens e.g. if we swallow only to the right and expose either no vertex or one white vertex). The second case is that we swallow to the left, and/or input one black vertex. If we perform the corresponding truncation on Y_n, and if necessary input independently a new bit (white or black with probability p or $1-p$), this preserves properties (a) and (b). The third possibility is that we reveal 3 edges and thus input to X_n either $\bullet\circ$ or $\bullet\bullet$ (read from right to left), with relative probabilites p and $1-p$. Again, this may straightforwardly be coupled with an input of two independent bits into Y_n so that property (b) is preserved. This proves the result. □

In the next result we let $(V_n)_{n\geq 0}$ denote an arbitrary sequence of i.i.d. random variables in $\mathbb{Z}^k$ such that V_n is independent of X_n for all n, and we let V have the same distribution as the V_n and be independent of $\xi^{(p)}$ and of ξ^*.

Lemma 3.6. *Let $F : \Sigma \times \mathbb{Z}^k \to \mathbb{R}$ and $F^* : \Sigma^* \times \mathbb{Z}^k \to \mathbb{R}$ be bounded and continuous functions. Consider the processes $(X_n)_{n\geq 0}$ and $(X^*)_{n\geq 0}$ started in the invariant distributions $\mu^{(p)}$ and μ^*, respectively.*

(1) *In the transient and null-recurrent cases,*

$$\frac{1}{n}\sum_{j=0}^{n-1} F(X_j, V_j) \to \mathbb{E}[F(\xi^{(p)}, V)] \quad \textit{almost surely.}$$

(2) *In the positive-recurrent case,*

$$\frac{1}{n}\sum_{j=0}^{n-1} F^*(X_j^*, V_j) \to \mathbb{E}[F^*(\xi^*, V)] \quad \textit{almost surely.}$$

Proof. In either case, the process $((X_n, V_n))_{n\geq 0}$ or $((X_n^*, V_n))_{n\geq 0}$ is a Markov process started in its unique invariant distribution. Hence the result follows from a standard ergodic theorem for Markov processes, see e.g. [17, Corollary 5.12]. □

4. The critical probability

Consider the peeling process started with the stationary version $\xi^{(p)}$ of the boundary. By Lemma 3.5, this boundary condition is amenable, and hence by Proposition 2.3 the critical probability is equal to $p_c = p_c^\bullet$; that is, the critical probability is the same as if we had started from an all-black boundary. We will now use this together with Proposition 2.2 and Lemma 3.6 to relate p_c to the function $\alpha(p)$ in Theorem 1.1.

4.1. **Proof of Theorem 1.1.** Let $(\zeta_n)_{n\geq 1}$ denote a sequence of i.i.d. random variables (independent of everything else) satisfying

$$\mathbb{P}(\zeta_n = 0) = 1 - p,\ \mathbb{P}(\zeta_n = 1) = p(1 - p),\ \text{and } \mathbb{P}(\zeta_n = 2) = p^2. \tag{4.1}$$

We identify ζ_n with the number of consecutive new white vertices from right to left (starting from the rightmost) on the revealed face f_n when $\mathcal{E}_n = 3$ (see Fig. 2). Let $(\chi_n)_{n\geq 1}$ be a sequence of i.i.d. random variables (independent of everything else) satisfying $\mathbb{P}(\chi_n = 0) = 1 - p$ and $\mathbb{P}(\chi_n = 1) = p$. We identify χ_n with the number of new white vertices on the revealed face f_n when $\mathcal{E}_n = 2$.

Let $(\hat{S}_n)_{n\geq 0} \in \mathbb{Z}$ be the process given by $\hat{S}_0 = 1$ and

$$\begin{aligned}\hat{S}_{n+1} = &\hat{S}_n + \mathbf{1}_{\{\mathcal{E}_n=3\}}\zeta_n + \mathbf{1}_{\{\mathcal{E}_n=2\}}\Big(\chi_n + \chi_n \sum_{k=1}^{\infty} \mathbf{1}_{\{X_n(k+1)=\circ;\mathcal{R}_n^-=k\}} - \mathcal{R}_n^+\Big) \\ &+ \mathbf{1}_{\{\mathcal{E}_n=1\}}\Big(\sum_{k=1}^{\infty} \mathbf{1}_{\{X_n(k+1)=\circ;\mathcal{R}_n^-=k\}} - \mathcal{R}_n^+\Big).\end{aligned} \tag{4.2}$$

Letting τ denote the minimal n for which $\hat{S}_n \le 0$, we have that $S_n = \hat{S}_{n\wedge\tau} \vee 0$. That is, S_n is obtained by running $\hat{S}_n$ until it hits $\{\dots,-2,-1,0\}$ and then freezing it at 0.

Let $V_n = (\mathcal{E}_n, \mathcal{R}_n^-, \zeta_n, \chi_n) \in \mathbb{Z}^4$. Note that $(V_n : n \ge 0)$ is an i.i.d. sequence, and that V_n is independent of X_n for each $n \ge 0$. We can write

$$\hat{S}_{n+1} = \hat{S}_n + F\big(X_n, V_n\big) - \mathcal{R}_n^+ \tag{4.3}$$

where

$$F(X_n, V_n) = \mathbf{1}_{\{\mathcal{E}_n=3\}}\zeta_n + \mathbf{1}_{\{\mathcal{E}_n=2\}}\chi_n + (\mathbf{1}_{\{\mathcal{E}_n=1\}} + \mathbf{1}_{\{\mathcal{E}_n=2\}}\chi_n)\sum_{k=1}^{\infty}\mathbf{1}_{\{X_n(k+1)=\circ;\mathcal{R}_n^-=k\}}.$$

Thus

$$n^{-1}\hat{S}_n = n^{-1} + n^{-1}\sum_{i=0}^{n-1} F\big(X_i, V_i\big) - n^{-1}\sum_{i=0}^{n-1}\mathcal{R}_i^+. \tag{4.4}$$

Note that $F(X_n, V_n) \le 4$ is bounded. Also note that, in the positive-recurrent case, we may equivalently write $F(X_n, V_n) = F^*(X_n^*, V_n)$ where

$$F^*(X_n^*, V_n) = \mathbf{1}_{\{\mathcal{E}_n=3\}}\zeta_n + \mathbf{1}_{\{\mathcal{E}_n=2\}}\chi_n + (\mathbf{1}_{\{\mathcal{E}_n=1\}} + \mathbf{1}_{\{\mathcal{E}_n=2\}}\chi_n)\sum_{k=1}^{W_n}\mathbf{1}_{\{X_n^*(k+1)=\circ;\mathcal{R}_n^-=k\}}.$$

Thus, applying Lemma 3.6 as well as the strong law of large numbers to (4.4), we deduce that for all p

$$n^{-1}\hat{S}_n \xrightarrow{\text{a.s.}} \mathbb{E}[F(\xi^{(p)}, V)] - \mathbb{E}(\mathcal{R}^+) = \alpha(p). \tag{4.5}$$

From (4.5) we see that if $\alpha(p) < 0$ then $\hat{S}_n \xrightarrow{\text{a.s.}} -\infty$, meaning that $S_n \xrightarrow{\text{a.s.}} 0$. Using Proposition 2.2 it follows that

$$p_{\mathrm{c}} \ge \sup\{p \in [0,1] : \alpha(p) < 0\}.$$

On the other hand, if $\alpha(p) > 0$ then $\hat{S}_n \xrightarrow{\text{a.s.}} \infty$. We claim that this implies that $\mathbb{P}(S_n > 0 \text{ for all } n) > 0$ and hence using Proposition 2.2 again that

$$p_{\mathrm{c}} \le \inf\{p \in [0,1] : \alpha(p) > 0\}.$$

To see the claim, first note that there is some N such that

$$\mathbb{P}(\hat{S}_n > 0 \text{ for all } n \ge N) > 0.$$

Also recall that the process $((X_n, \hat{S}_n))_{n\ge 0}$ is a Markov chain. Fix a sample of X_0 and a sequence of peeling moves $\pi = (\pi_0, \pi_1, \dots, \pi_{N-1}, \pi_N, \dots)$ such that $\hat{S}_n > 0$ for all $n \ge N$ (the π_j encode which face is discovered and what the colours of the new vertices are). We show that there are peeling moves $\pi_0', \pi_1', \dots, \pi_{N-1}'$ such that if we instead perform the sequence $\pi' = (\pi_0', \pi_1', \dots, \pi_{N-1}', \pi_N, \pi_{N+1}, \dots)$ then (i) for all $n \le N$, the X_n are the same as if we had performed the sequence π, (ii) $\hat{S}_{n+1} - \hat{S}_n \ge 0$ for all $n \le N-1$, and (iii) $\hat{S}_N$ is at least as large as if we had performed the sequence π. Moreover, the π_n' can be chosen so that all the ratios $\mathbb{P}(\pi_n')/\mathbb{P}(\pi_n)$ are uniformly bounded from below by a positive number, for all choices of the π_n. Once we show that there are such π_n' the claim follows.

We describe how to choose π_n' given π_n in a case-by-case manner. If $\mathcal{R}_n^+ = 0$ then we just take $\pi_n' = \pi_n$. Assume in what follows that $\mathcal{R}_n^+ > 0$, and to start with also that $\mathcal{R}_n^- = 0$. If π_n exposes no vertex, or exposes exactly one vertex which is white, then π_n does not change X_n. In this case let π'_n be given by exposing two vertices, both white. Then π'_n also does not change X_n, and $\mathbb{P}(\pi'_n)/\mathbb{P}(\pi_n) \geq \mathbb{P}(\pi'_n) = \frac{3}{8}p^2$. The next case is that π_n exposes exactly one vertex which is black, so that X_{n+1} is obtained by inputting one black vertex to the front of X_n. We then let π'_n be given by exposing two vertices, the first white and the second black (counting counter-clockwise). Now $\mathbb{P}(\pi'_n)/\mathbb{P}(\pi_n) \geq \mathbb{P}(\pi'_n) = \frac{3}{8}p(1-p)$. The final case is when π_n is given by swallowing k edges to the left and ℓ edges to the right. Then let π'_n be given by swallowing k vertices to the left only, and exposing one white vertex. In this case we have, using (1.5),

$$\frac{\mathbb{P}(\pi'_n)}{\mathbb{P}(\pi_n)} = \frac{pq_k}{q_{k,\ell}} = \frac{pq_k}{\frac{8}{3}q_k q_\ell} \geq \tfrac{3}{8}p. \tag{4.6}$$

This proves the claim and hence the theorem. □

4.2. **Upper and lower bounds.** Although we are unable to explicitly find the probabilites $\mathbb{P}(\xi^{(p)}(k) = \circ)$, and hence the function $\alpha(p)$, we can find upper and lower bounds. As a warmup, we prove Proposition 1.2. Trivially $\beta(p) \geq 0$ for all p, and this already gives $p_c \leq \frac{\sqrt{73}-5}{6}$. Moreover, from Lemma 3.5 we have that $\mathbb{P}(\xi^{(p)}(k) = \circ) \leq p$ for all k (since the event $\{\xi(k) = \circ\}$ is increasing). From this and the fact (2.2) that $\sum_{k\geq 1,\mathrm{odd}} q_k = \frac{1}{8}$ and $\sum_{k\geq 1,\mathrm{even}} q'_k = \frac{1}{18}$ we deduce that $p_c \geq \frac{\sqrt{493}-13}{18}$. This proves Proposition 1.2.

We will now define a Markov chain with finite state space that will allow us to improve these bounds. To do this, it helps to first recall the process $X^*_n = (X^*_n(k) : 1 \leq k \leq W_n) \in \hat{\Sigma}^*$, in particular the fact that $X^*_n(k) = X_n(k)$ for all $k \leq W_n$. Thus the 'flag' W_n keeps track of where in X_n we can find X^*_n. For each $K \geq 1$, we will define a process

$$X^K_n = (X^K_n(k) : 1 \leq k \leq W^K_n) \in \hat{\Sigma}^K.$$

Here $\hat{\Sigma}^K$ is the set of sequences in $\hat{\Sigma}^*$ of length at most K and W^K_n is the length of X^K_n. We start with $X^K_0 = \bullet$, and thus $W^K_0 = 1$. Supposing we have defined X^K_n and X_n for some $n \geq 0$, we look at the next peeling move of X_n. If we 'swallow beyond X^K_n', that is $\mathcal{R}^- \geq W^K_n - 1$, set $X^K_{n+1} = \bullet$. Otherwise we first apply the usual rules to X^K_n, and then (if necessary) truncate at K to obtain X^K_{n+1} which satisfies $W^K_{n+1} \leq K$. Since X^K_{n+1} depends on X^K_n and the independent randomness in the next peeling move of X_n, it follows that $(X^K_n)_{n\geq 0}$ is a Markov chain. Moreover, we have coupled X^K with X and X^* in such a way that for all n,

$$W^K_n \leq W_n \wedge K, \text{ and } X^K_n(k) = X^*_n(k) = X_n(k) \text{ for all } k \leq W^K_n. \tag{4.7}$$

It is not hard to see that X^K is an aperiodic and irreducible Markov chain, and thus has a unique asymptotic distribution μ^K. The transition probabilities for this chain may be written down explicitly, and hence also (at least in principle) the measure μ^K.

Define $X^{K,\bullet}_n$ by concatenating X^K_n with an infinite sequence of $\bullet$'s. If we start X in the all-black state, then (4.7) implies that $X_n \geq X^{K,\bullet}_n$ for all n. Moreover, the distribution of $X^{K,\bullet}_n$ converges weakly, as $n \to \infty$, to a measure $\mu^{K,\bullet}$ which may be obtained from μ^K in a straightforward way, and which satisfies $\mu^{K,\bullet} \leq \mu^{(p)}$. Thus we may use $\mu^{K,\bullet}$ to obtain a lower bound on $\alpha(p)$ and hence an upper bound on p_c. We may similarly define a process $X_n^{K,\circ}$ by concatenating X_n^K with an infinite sequence of $\circ$'s, and thus obtain a measure $\mu^{K,\circ}$ satisfying $\mu^{(p)} \leq \mu^{K,\circ}$. However, there is another, better, way to obtain an upper bound on $\mu^{(p)}$, as follows.

Recall from Lemma 3.5 that we coupled X to a chain Y such that for each n the distribution of Y_n is $\textsc{iid}(p)$, and $X_n \leq Y_n$. Define $X_n^{K,\textsc{iid}(p)}(k)$ to be $X_n(k) = X_n^K(k)$ if $k \leq W_n^K$, or $Y_n(k)$ otherwise. Thus we have for all n that $X_n^{K,\textsc{iid}(p)} \geq X_n$. Moreover, the distribution of $X_n^{K,\textsc{iid}(p)}$ converges weakly, as $n \to \infty$, to a measure $\mu^{K,\textsc{iid}(p)}$ which may be obtained by first sampling $\xi^K \in \hat{\Sigma}^K$ from μ^K and then appending to it an infinite $\textsc{iid}(p)$ sequence (independent of ξ^K). It follows that $\mu^{K,\textsc{iid}(p)}$ stochastically dominates $\mu^{(p)}$.

As an example, taking $K = 2$ the relevant states of X^2 are $\bullet$, $\bullet\bullet$ and $\circ\bullet$ (each state read from right to left). Using (2.2)–(2.4) we find that the transition probabilities are:

Transition	Probability
$\bullet \to \bullet\bullet$	$\frac{1}{2}(1-p)$
$\bullet \to \circ\bullet$	$\frac{3}{8}p(1-p)$
$\bullet\bullet \to \bullet$	$\frac{1}{9}(1+p)$
$\bullet\bullet \to \circ\bullet$	$\frac{3}{8}p(1-p)$
$\circ\bullet \to \bullet$	$\frac{1}{9}(1+p)$
$\circ\bullet \to \bullet\bullet$	$\frac{1}{2}(1-p)$

(4.8)

and hence

$$\mu^2(\bullet) = \frac{8(1+p)}{-27p^2 - p + 44} \tag{4.9}$$

$$\mu^2(\circ\bullet) = \frac{27p(1-p)}{-27p^2 - p + 44}. \tag{4.10}$$

Using that $\mu^{2,\bullet}(\circ\bullet) = \mu^2(\circ\bullet)$ and $\mu^{2,\textsc{iid}(p)}(\circ\bullet) = p\mu^2(\bullet) + \mu^2(\circ\bullet)$ as well as (1.8) and (2.2) one finds that a lower bound on p_c is given by the unique solution in $[0,1]$ to

$$189p^4 + 378p^3 - 596p^2 - 575p + 396 = 0 \tag{4.11}$$

and an upper bound is given by the unique solution in $[0,1]$ to

$$81p^4 + 162p^3 - 251p^2 - 232p + 176 = 0. \tag{4.12}$$

The result is $0.523599 \leq p_c \leq 0.572542$ when rounding to six digits.

One may similarly write down the transition probabilities for X^K for general K, but as K becomes larger it quickly becomes infeasible to write down the limiting distributions $\mu^{K,\bullet}$ and $\mu^{K,\textsc{iid}(p)}$ by hand. We provide in Table 1 some upper and lower bound which we numerically computed using this method, and Fig. 3 shows a plot of these bounds for K in the range 2 to 17. We note that the gap between the bounds we obtain is (weakly) decreasing in K due to stochastic monotonicity of the measures $\mu^{K,\bullet}$ and $\mu^{K,\textsc{iid}(p)}$ in K.

K	Lower bound	Upper bound
4	0.5382	0.5656
6	0.5436	0.5625
8	0.5464	0.5609
10	0.5482	0.5598
12	0.5493	0.5591
14	0.5502	0.5586
16	0.5508	0.5583
17	0.5511	0.5581

TABLE 1. Upper and lower bounds on p_c, obtained by numerically finding the limiting distributions for the processes $X_n^{K,\text{IID}(p)}$ and $X^{K,\bullet}$.

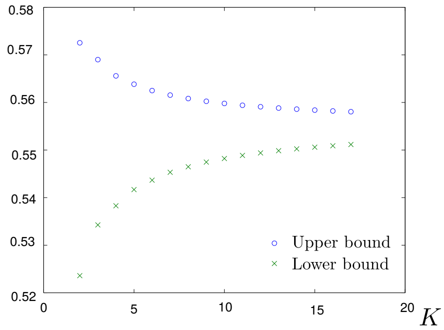

FIGURE 3. A plot of upper bounds and lower bounds on p_c as a function of K.

5. Outlook

Certain questions are left unanswered by this work, the most obvious one being what the exact value of p_c is? It may be possible to find the exact distribution of $\xi^{(p)}$, and thereby presumably also p_c, but this would require a new idea. We also do not provide much information about general properties of the function $\alpha(p)$, for example we have not showed that it has a *unique* root in $[0, 1]$, which seems natural to suppose. A related question is whether the processes $(X_n)_{n\geq 0}$ and $(S_n)_{n\geq 0}$ are stochastically monotonic in p? This also does not seem easy to establish.

On the other hand, the methods we have presented should (at least in principle) not be hard to extend to site percolation on other half-planar maps with the domain Markov property. Angel and Ray showed in [5] that for each $k \geq 3$ there is a one-parameter family of translation-invariant, domain Markov probability measures supported on half-planar k-angulations with only simple faces. Extending the methods to the full class of such *quadrangulations* appears straightforward: one need only adjust the values of the q:s. The methods should in principle also extend to $k \geq 5$. Then the notion of admissible boundary conditions would need to be modified to allow for segments of consecutive white vertices of length up to $k - 3$, and the formulas would become considerably more complicated, but there does not seem to be any fundamental problem.

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