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Topics in Random Tree Theory

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Abstract

In this thesis we study two models of random trees. The first model is an equilibrium statistical mechanical model of trees where the action is given by a sum over the vertices and depends only on their degrees. We consider two classes of such trees: *caterpillars* which are trees with the property that all vertices of degree greater than one form a simple path, and *planar trees*, which are often referred to as *branched polymers*. The purpose of this study is to characterize the phase structure of the models with a special emphasis on describing a phase where a vertex of infinite degree emerges in the thermodynamic limit. We show that both classes of trees exhibit two phases, an *elongated phase* and a *condensed phase*. We prove convergence of the finite volume Gibbs measures to a measure on the set of infinite trees. In the elongated phase the measure is concentrated on the set of trees with exactly one path from a given vertex to infinity and in the condensed phase it is concentrated on the set of trees with exactly one vertex of infinite degree. We conclude the discussion of each class by calculating the Hausdorff and spectral dimension in both phases.

The second model we consider is a new model of growing random trees, referred to as the *vertex splitting model*. In each time step, the trees are grown by selecting a vertex and splitting it into two vertices which are joined by a new edge. The model reduces, in special cases, to the preferential attachment model, Ford's alpha model for phylogenetic trees and its generalization the alpha–gamma model. We develop a mean field theory for the vertex degree distribution, prove that the mean field theory is exact in some special cases and check that it agrees with numerical simulations in general. We construct certain correlation functions which enable us to calculate the Hausdorff dimension of the trees. The Hausdorff dimension depends on the parameters of the model and can vary from one to infinity. We study correlations between degrees of neighbouring vertices and compare the result to graphs where no correlations are present. We conclude by showing how the vertex splitting model is related to other models of random trees and provide new results on the alpha model.

Ágrip (in Icelandic)

Við rannsökum tvö líkön af slembitrjám. Fyrra líkanið er safneðlisfræðilíkan trjáa þar sem orkan er gefin með summu yfir hnúta og er einungis háð stigi þeirra. Við skoðum tvö söfn trjáa: *margfætlur*, sem eru tré með þann eiginleika að allir hnútar af stigi hærra en einn mynda einfaldan veg og *sléttutré*. Tilgangur rannsóknarinnar er að lýsa mismunandi fösum líkansins og sérstök áhersla er lögð á að lýsa fasa þar sem hnútur af óendanlegu stigi verður til þegar stærð trjánna stefnir á óendanlegt. Við sýnum að bæði söfnin hafa tvo fasa sem við köllum *langan fasa* og *þéttan fasa*. Við sönnum að Gibbs málin, á endanlegum mengjum trjáa, eru samleitin og stefna á mál á mengi óendanlegra trjáa. Í langa fasanum hefur málið stoð á mengi trjáa sem innihalda nákvæmlega einn veg frá gefnum hnútpunkti út í óendanlegt en í þétta fasanum hefur málið stoð á mengi trjá sem innihalda nákvæmlega einn hnútpunkt af óendanlegu stigi. Við ljúkum umræðunni um hvort safn fyrir sig með því að reikna Hausdorff- og litrófsvídd beggja fasa.

Síðara líkanið sem við skoðum er nýtt líkan af vaxandi slembitrjám sem við nefnum *hnútaskiptingalíkanið*. Í hverju tímaskrefi, vaxa tréin með þeim hætti að hnútpunktur er valinn af handahófi og honum skipt í tvo hnúta sem tengdir eru með nýjum legg. Líkanið inniheldur sem sértilfelli viðhengilíkanið, alfalíkan Fords af þróun tegunda og alhæfingu þess alfa–gammalíkanið. Við reiknum hnútastigsdreifingu í stórum trjám með meðalsviðsfræði, sönnum réttmæti meðalsviðsfræðinnar í sértilfellum en styðjum almenna tilvikið með tölvuhermunum. Við smíðum ákveðin fylgniföll sem gera okkur kleift að reikna Hausdorffvídd trjánna. Hausdorffvíddin er háð stikum líkansins og getur tekið öll gildi á bilinu einn upp í óendanlegt. Við reiknum fylgni milli hnútastigs næstu nágranna og berum saman við net þar sem engin fylgni er til staðar. Við ljúkum ritgerðinni með því að bera hnútaskiptingalíkanið saman við önnur líkön af slembitrjám og sönnum m.a. nýjar niðurstöður um alphalíkan Fords.

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1

Introduction

1.1 Background

Random graphs are used in many branches of science to describe relationships between various entities and to model physical objects. The former case includes social networks [2], phylogenetic trees [5,43,45], the world-wide web [3] and much more. The latter case includes discrete objects such as macromolecules [33] and branched polymers [2]. The graphs can also serve as a mathematical tool to approximate continuous objects, an example of this being triangulations of manifolds in quantum gravity, see e.g. [8].

Random trees are random graphs which have, for example, been used to model family trees and evolving populations [46], phylogenetic trees [5, 43, 45], fragmentation and coagulation processes [16] and more. Trees are also of theoretical importance in the research of more general random graphs. Planar trees, for instance, encode information on folded RNA molecules through their connection with planar arch structures [33] and labelled trees are important in the study of the statistics of planar maps and their scaling limits [56,57] via Schaeffer's bijection [61]. Planar maps are a prominent tool in one approach to quantum gravity. A simple special case of Schaeffer's bijection is given in [39] between generic trees and 2D causal dynamical triangulations [11].

In this thesis we study two types of models of random trees. In Part I we consider an equilibrium statistical mechanical model of two different classes of trees where the action is given by a sum over the vertices of a tree and depends only on their degrees.

The first class of trees we consider are so–called *caterpillars* which are trees with the property that all vertices of degree greater than one form a simple path. The second class we consider contains all planar trees, which are often referred to as *branched polymers*. The purpose of this study is to give a complete characterization of the phase structure of the models with a special emphasis on describing a phase where vertices of infinite degree emerge in the thermodynamic limit. The main motivation is to solve the model of branched polymers and the model of caterpillars is a step towards that solution. Both models have been analysed extensively before; although the caterpillar model has usually been studied in a different context.

The equilibrium statistical mechanical model of branched polymers was introduced by Meir and Moon in 1978 [58] under the name *simply generated trees*. They derived the asymptotic behaviour of the finite volume partition function under certain assumptions on the parameters of the model. The model, as considered in the present context, was first studied by Ambjørn et al. in the late 1980's in the papers [6, 7]. In 1996, Bialas and Burda calculated the critical exponents in the model and described its phase structure [18]. They argued that the model exhibits two phases in the thermodynamic limit: a *fluid (elongated, generic) phase* where the trees are of a large diameter and have vertices of finite degree and a *condensed (crumpled) phase* where the trees are short and bushy with exactly one vertex of infinite degree. In 2007, Durhuus, Jonsson and Wheater gave a complete characterization of the fluid phase, referred to as *generic trees*, by showing that the Gibbs measures converge to a measure concentrated on the set of trees with exactly one path to infinity with finite critical Galton—Watson outgrowths [38]. They furthermore proved that the trees have a Hausdorff dimension $d_H = 2$ and a spectral dimension $d_s = 4/3$ with respect to the infinite volume measure. The main goal of Part I in this thesis is to establish analogous results for the condensed phase. Preliminary results in this direction were obtained by the author in [64].

One of the motivations for the study of the branched polymer model is that a similar phase structure is seen for more general classes of graphs in models of simplicial gravity [1,9]. In these models the elongated phase is effectively described by trees [10] and it has been established by numerical methods that in the condensed phase a single large simplex appears whose size increases linearly with the graph volume [27,48]. A closely related phenomenon of condensation also appears in dynamical systems such as the zero range process [40].

In 1997, Bialas et al. proposed [23] that the same mechanism is behind the phase transition in the different models and the so–called *constrained mean field model* was introduced in order to capture this feature. The authors followed up on the idea in a series of papers [17, 19–22] where the model was studied under the name *balls in boxes* or *backgammon model*. The model consists of placing N balls into M boxes and assigning a weight to each box depending only on the number of balls it contains. This model is closely related to the equilibrium statistical mechanical model of caterpillars. In [20] the critical exponents of the balls in boxes model were calculated and the two phases characterized. The distribution of the box occupancy number was derived and it was argued that in the condensed phase exactly one box contains a large number of balls which increases linearly with the system size.

The model of caterpillars was studied by the author, in collaboration with Þórður Jónsson, in a recent paper [52]. Our original motivation for studying the model was that, despite its simplicity, it was predicted to have the same phase structure as the more complicated model of branched polymers. The caterpillar model was solved by proving convergence of the Gibbs measures to a measure on infinite graphs and a complete characterization of the limiting measure was provided in both phases. We believe this to be the first rigorous treatment of the condensed phase in models of the above type and it guided us towards a solution of the condensed phase in the model of branched polymers. A model of random combs, equivalent to the caterpillar model was studied in [36] where analogous results were obtained for the limiting measure.

In Part II we consider a new model of randomly growing planar trees referred to as the *vertex splitting model*. This work is based on and extends the paper [32] which is a joint work of the author with François David, Mark Dukes and Þórður Jónsson. We will describe the model informally here, a precise definition will be given in Part II. The parameters of the model are nonnegative weights $w_{i,j}$ and a probability measure is generated on the set of finite trees by the following growth rule. Start from an initial tree T_0. Select a vertex in T_0 of degree k with a probability weight

$$w_k = \frac{k}{2}\sum_{n=1}^{k+1} w_{n,k+2-n}$$

and "split" the vertex into two new vertices of degree i and j with a probability weight $w_{i,j}$, see Fig. 1.1. The numbers w_k are referred to as *splitting weights* and the numbers $w_{i,j}$ are referred to as *partitioning weights*. This process is repeated indefinitely. In most cases we will put an upper bound, D, on the degrees of vertices by choosing $w_{D+1,1} = w_{1,D+1} = 0$ such that vertices of degree $D+1$ are not produced by the process. The model originates from a slightly different model of growing planar

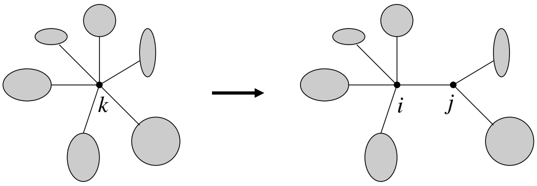

Figure 1.1: The splitting operation.

trees, introduced in 2008 by David, Hagendorf and Wiese in connection with RNA folding [33]. We will describe this relationship in a few words but refer to [33] for a more detailed explanation.

A completely folded (planar) RNA molecule is represented by a system of non intersecting arches which connect sites $1, \ldots, n$ on a line in such a way that no more sites can be connected without crossing arches. The sites represent the bases in the molecule and the arches represent the pairing of bases. These structures are in one to one correspondence with planar trees having vertices of two types: a grey vertex and a white vertex decorated with an arrow. The faces of the arches correspond to the vertices of the tree and the arches correspond to the edges of the tree. If a face is adjacent to an unpaired base the colour of the corresponding vertex is white and an arrow points from the vertex to the unpaired base, otherwise the vertex is grey, see Fig. 1.2.

The model of the RNA folding process in [33] is defined by starting with a strand of n unpaired sites and depositing arches uniformly at random to the sites such that no arches cross. This is repeated until no more arches can be added. The arch deposition model can equivalently be described by the following growth process. Start with an empty strand at time zero and in each time step add a site to the strand, its location chosen uniformly at random, and pair the new site if it is possible without crossing

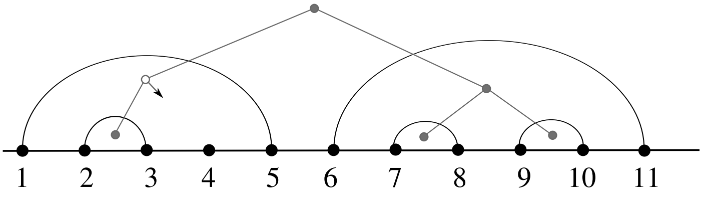

Figure 1.2: The connection between arch structures and decorated trees.

arches. By viewing the evolution of the corresponding tree one finds that it changes in one of two ways: a grey vertex turns white and is decorated with an arrow (the new site is not paired) or an additional arrow is placed on a white vertex which is then split into two grey vertices connected by a new edge which is orthogonal to the two arrows (the new site is paired), see Fig. 1.3. The splitting of the white vertices in the above growth process is essentially the same operation as in the vertex splitting model. In the RNA model the splitting weights are fixed by the dynamics of the arch deposition process.

The vertex splitting model has very general growth rules and includes other previously studied random tree models. It becomes a special case of the preferential attachment growth model, also referred to as *random recursive trees*, (see e.g. [2,31]) when we take $w_{j,k} = 0$ unless j or k is equal to 1. It also has, as a limiting case, Ford's alpha model of phylogenetic trees [43] and its generalization, the alpha–gamma model [28].

Our main motivation is to develop general tools to study the properties of models of random tree growth. In particular we are motivated by the issues of unification and of universality: Is there a general tree growth process which can encompass the different models which are known at the moment? How many different universality classes, i.e. continuous tree models with different scaling properties (exponents and correlation functions), exist in this framework? The results presented here are a first step in this direction.

1.2 Outline

Chapter 2 of Part I is based on the paper [52], written in collaboration with Þórður Jónsson. We solve the equilibrium statistical mechanical model of caterpillars by proving convergence of the Gibbs measures to a measure on infinite graphs and give a complete characterization of the limiting measure. We show that in the fluid phase

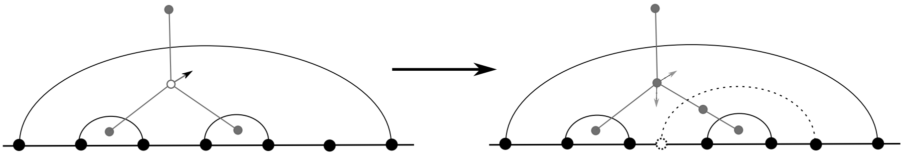

Figure 1.3: A step in the RNA growth process.

the measure is concentrated on the set of caterpillars having infinite length and that in the condensed phase it is concentrated on the set of caterpillars which are of finite length and have precisely one vertex of infinite degree. We conclude Chapter 2 by calculating the Hausdorff dimension and spectral dimension of the caterpillars in both phases and at the phase transition.

In Chapter 3 of Part I we study the equilibrium statistical mechanical model of branched polymers. This work is based on the paper [53] (in preparation) written in collaboration with Þórður Jónsson. We generalize the definition of planar trees in [35] to allow for vertices of infinite degree and define a new metric on this set of planar trees. This new metric space is compact and the subset of finite trees is dense. We use similar techniques as for the caterpillar model to prove convergence of the Gibbs measures in both phases with respect to this metric. We prove that in the condensed phase the limiting measure is concentrated on the set of trees of finite diameter with precisely one vertex of infinite degree and that the rest of the tree is distributed as a subcritical Galton—Watson process with mean offspring probability $m < 1$. Furthermore, we prove that in finite trees the degree of the large vertex grows linearly with the system size, N, as $(1-m)N$ with probability arbitrarily close to one, confirming the result stated in [26] . We conclude by calculating the spectral dimension of the infinite measure in the condensed phase. In [29] it was claimed, on the basis of scaling arguments, that the spectral dimension is $d_s = 2$. We prove, however, that if the spectral dimension exists it depends continuously on a parameter of the model and can take any value greater than two. In fact, it takes the same values as the spectral dimension of the condensed phase in the caterpillar model.

Part II is based on and extends the paper [32], written in collaboration with François David, Mark Dukes and Þórður Jónsson. We first give the precise definition of the vertex splitting model. We then study the special case where the splitting probability weights are linear with the initial vertex degree i and focus on the vertex degree distribution. In Chapter 6 we write exact recurrence equations for the general local vertex degree probability distributions. Using the Perron-Frobenius theorem [62] we show that the single vertex degree probability distribution $\boldsymbol{\rho} = \{\rho_k\}$ (ρ_k is the density of vertices with a given degree k) has a well defined limit as the size of the tree goes to infinity which is independent of the initial tree. We furthermore show that $\boldsymbol{\rho} = \{\rho_k\}$ is given by an eigensystem equation of the form $\mathbf{B}\boldsymbol{\rho} = \lambda\boldsymbol{\rho}$, where $\mathbf{B}$ is a matrix depending on the weights of the model. The proof depends on the matrix $\mathbf{B}$ being diagonalizable. Similar techniques have been used to find the asymptotic degree distribution in random recursive trees [49].

In Section 6.4 we relax the condition of linearity on the splitting weights w_i. We argue that mean field theory is still valid and that the degree probability distribution $\boldsymbol{\rho}$ is still given by the same linear eigensystem equation as in the linear case. We give good numerical evidence of the validity of these mean field equations for $D = 3$ trees. For infinite D and linear and uniform splitting probabilities we can still calculate the vertex degree distribution in closed form using mean field theory. This is done in section 6.5, where we show that it agrees with numerical simulations. The vertex degree distribution is found to fall off factorially in this case.

In Chapter 7 we study probabilities associated to the local subtree structure of the tree, as seen from any vertex, and as a function of its creation time s. More precisely, we are able to write recursion relations for the probability $p_k(\ell_1, \cdots, \ell_k; s)$ that the vertex created at time s is of degree k, with the k subtrees with fixed respective sizes $\ell_1, \cdots, \ell_k$. These subtree structure probabilities are related to the radius of the tree and their scaling properties allow us to extract the Hausdorff dimension of the trees. Using a natural scaling hypothesis, we show that the Hausdorff dimension d_H is given by the solution of an eigensystem equation of the form $\mathbf{C}\boldsymbol{\omega} = w_2/d_H\, \boldsymbol{\omega}$, where $\mathbf{C}$ is a matrix which is a function of the weights of the model. We use a Perron-Frobenius argument to prove that this eigensystem equation has a unique physical solution. We establish bounds on the Hausdorff dimension and show that it can vary continuously with the splitting weights between 1 and $+\infty$.

In Chapter 8 we study the correlations between the degrees of neighbouring vertices. This amounts to studying the density ρ_{ij} of links with vertices of degrees i and j. We write general equations for these correlations in the linear splitting weight case. In the simple case of $D = 3$ trees these correlations are calculated explicitly, and compared with numerical simulations. In Section 8.3 we extend our results for the case of non–linear splitting weights, assuming mean field theory. We show that there is a very good agreement between our analytical results and numerical simulations. We conclude by discussing the amount of assortative mixing in the vertex splitting model i.e. whether vertices of high degree prefer to be neigbours of vertices of high degree or to be neighbours of vertices of low degree. For recent research on assortative mixing in networks we refer to [54, 59, 67].

In the final chapter we discuss in more detail the relationship between our model and other models of random trees, in particular the alpha model of phylogenetic trees. We prove convergence of the finite volume measures generated by the growth rules of the alpha model and calculate the annealed Hausdorff dimension. This work is based on and extends the paper [65] by the author.

1.3 Definitions

In this section we collect together the basic definitions and notation which are used in the following chapters.

1.3.1 Graphs

A graph G is composed of a vertex set $V(G)$ and an edge set $E(G)$ which consists of unordered pairs $\{v, w\}$ where $v, w \in V$ and $v \neq w$ [1]. In the following we will always single out a vertex in V called the *root* of the graph and denote it by r. The number of elements in a set A is denoted by $|A|$. We define $|G| \equiv |E|$ and call it the *volume* of the graph G. Two vertices $v, w \in V$ are called *neighbours* if $\{v, w\} \in E$. The number of neighbours of a vertex $v \in V$ is called the *degree* or *order* of v and is denoted by $\sigma_G(v)$ or simply $\sigma(v)$ if it is clear to which graph the vertex belongs. A common constraint on graphs is that all vertices have finite degree in which case the graph is said to be locally finite. However, here we sometimes allow the possibility that graphs have vertices of infinite degree.

A finite path γ in a graph G is a finite sequence

$$\gamma = (v_0, v_1, v_2, \dots, v_n) \tag{1.1}$$

where $v_0, \dots, v_n \in V$ and v_{i-1} and v_i are neighbours for all $i = 1, \dots, n$. We call n the *length* of the path γ and define $|\gamma| = n$. The vertices v_0 and v_n are called the *endpoints* of γ. If $v_0 = v_n$ then the path is called a *cycle*. We say that a graph G is connected if for every two vertices $v, w \in V$, there exists a path between v and w. The graph distance d_G between any two vertices $v, w \in V$ in a connected graph G is defined by

$$d_G(v, w) = \min\{|\gamma| \ : \ \gamma \text{ a path with endpoints } v \text{ and } w\} \tag{1.2}$$

and $d_G(v, v) = 0$ for all $v \in V$.

A graph is called a *tree* if it contains no cycles. In this thesis we restrict our attention to connected trees, which are the objects of the models presented. We will put the additional condition of planarity on the trees under consideration, which means in words that the edges incident on a given vertex are ordered. This statement is ambiguous if the vertex has an infinite degree and therefore we give a proper

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[1] Some authors refer to these graphs as “simple graphs” and allow general graphs to have “self links” and “multi links”.

definition in Chapter 3. The condition of planarity is not essential in the models we consider, but is rather a convention, and in some cases it is motivated by the nature of the model. By convention we assume, unless otherwise stated, that the trees have a root of degree one. We denote the set of all connected, rooted planar trees, finite or infinite, by Γ and the set of all finite, connected, rooted planar trees by Γ'. The set of connected, rooted planar trees with N edges will be denoted by Γ_N. In trees, we denote the unique shortest path between vertices v and w by (v, w).

1.3.2 Random graphs

Let $\mathcal{G}_N$ be some subset of the set of graphs with N edges and let $\mathcal{G}$ be the set of all finite and infinite graphs of the same type. We will not be precise for the moment about how $\mathcal{G}$ is constructed. A random graph is defined by a probability distribution ν_N on $\mathcal{G}_N$. Usually the distribution ν_N is either (a) constructed explicitly for a given N or (b) defined in a recursive way from ν_{N-1}, commonly by a growth process.

The models in Part I are of type (a). They are defined by a set of nonnegative numbers $w_1, w_2, \ldots$, called *branching weights*, and a probability is assigned to a graph $G \in \mathcal{G}_N$ by

$$\nu_N(G) = \frac{1}{Z_N} \prod_{v \in V(G)} w_{\sigma_G(v)} \tag{1.3}$$

where

$$Z_N = \sum_{G' \in \mathcal{G}_N} \prod_{v \in V(G')} w_{\sigma_{G'}(v)} \tag{1.4}$$

is a normalization factor which is called the *finite volume partition function*. These models are referred to as *equilibrium statistical mechanical models* and ν_N is called the *Gibbs measure*. We say that the model has a *local action*, since the energy of a given graph is the sum over the independent energies of individual vertices in the graph.

The models in Part II are of type (b). They are defined by a growth rule, and the probability of a graph $G \in \mathcal{G}_N$ is given by

$$\nu_N(G) = \sum_{G' \in \mathcal{G}_{N-1}} \nu_{N-1}(G')\mathbb{P}(G' \to G) \tag{1.5}$$

where $\mathbb{P}(G' \to G)$ is the probability of growing the graph G from G' according to the growth rule.

In both the above cases we study properties of the graphs when $N \longrightarrow \infty$. In some cases it is possible to show that the measures ν_N, viewed as measures on $\mathcal{G}$, converge in a weak sense, to a measure ν which is concentrated on the set of infinite graphs. This is referred to as taking the infinite volume limit. In other cases one can only study convergence of some observables (or their expectation values), such as the vertex degrees, the graph diameter etc. In order to define the notion of weak convergence we assign a metric d to the set $\mathcal{G}$. The statement

$$\nu_N \longrightarrow \nu \tag{1.6}$$

in a weak sense as $N \longrightarrow \infty$ means that

$$\int_{\mathcal{G}} f d\nu_N \longrightarrow \int_{\mathcal{G}} f d\nu \tag{1.7}$$

as $N \longrightarrow \infty$ for all bounded functions f which are continuous in the metric d. The problem of taking the infinite volume limit involves, among other things, defining the set $\mathcal{G}$ properly and defining a suitable metric on $\mathcal{G}$.

We will use repeatedly the following result about weak convergence of probability measures which is stated in [35] and derived e.g. in [24]. If ν_N is a sequence of probability measures on a metric space $(\mathcal{G}, d)$ and $\mathcal{U}$ is a family of both open and closed subsets of $\mathcal{G}$ such that

(i) any finite intersection of sets in $\mathcal{U}$ are in $\mathcal{U}$,

(ii) any open subset of $\mathcal{G}$ may be written as a finite or a countable union of sets from $\mathcal{U}$ and

(iii) the sequence $\nu_N(A)$, converges as $N \longrightarrow \infty$ for all sets $A \in \mathcal{U}$,

then the sequence ν_N converges weakly provided it is tight, i.e. for any $\epsilon > 0$ there exists a compact subset C of $\mathcal{G}$ such that

$$\nu_N(\mathcal{G} \setminus C) < \epsilon \quad \text{for all } N. \tag{1.8}$$

The last condition of tightness is automatically fulfilled if the metric space is compact, which is the case in all applications in this thesis.

1.3.3 Random walks on graphs and the spectral dimension

A simple random walk on a graph G is a path ω together with a probability weight

$$\prod_{t=0}^{|\omega|-1} \left(\sigma_G(\omega_t)\right)^{-1} \tag{1.9}$$

where ω_t denotes the $(t+1)$-st vertex of ω. We think about the random walk as a process where at time t a walker, located at ω_t, moves to one of its neighbours with probabilities $\left(\sigma_G(\omega_t)\right)^{-1}$.

We begin by defining the *spectral dimension* of a graph which is loosely speaking, the dimension experienced by a random walker travelling on the graph. Let $p_G(t)$ be the probability that a simple random walk which begins at the root in G, is located at the root at time t. The spectral dimension of the graph G is defined as d_s provided that

$$p_G(t) \asymp t^{-d_s/2} \tag{1.10}$$

where we write $f(t) \asymp t^{-\gamma}$ if

$$\lim_{t\to\infty} \frac{\log\left(f(t)\right)}{\log(t)} = -\gamma. \tag{1.11}$$

If $p_G(t)$ falls off faster than any power of t then we say that $d_s = \infty$. This definition only makes sense on infinite graphs since on finite graphs, the return probability is asymptotically a positive constant. It is straightforward to verify that the spectral dimension of a connected, locally finite graph is independent of the choice of a root. The spectral dimension of the d–dimensional hyper–cubic lattice $\mathbb{Z}^d$ is $d_s = d$ in which case it agrees with our usual notion of dimension. For general graphs the spectral dimension need not be an integer and furthermore it might not exist. The spectral dimension can also be defined on a continuous manifold in which case it is the rate of decay of the heat kernel at coinciding points.

For an infinite random graph $(\mathcal{G}, \nu)$, where ν is a probability distribution on some class of graphs $\mathcal{G}$, one can define the spectral dimension in different ways. First of all the graphs can have, ν almost surely, a spectral dimension d_s defined as above. The statement that an event E happens ν almost surely means that $\nu(E) = 1$. Secondly we define the *annealed spectral dimension* as $\bar{d}_s$ provided that

$$\langle p_G(t)\rangle_\nu \asymp t^{-\bar{d}_s/2}. \tag{1.12}$$

where $\langle \cdot \rangle_\nu$ denotes expectation value with respect to ν. These definitions need not agree and in this thesis we encounter examples where $\bar{d}_s$ exists and is finite, whereas d_s is almost surely infinite.

The return probabilities are most conveniently analysed through their generating functions. In the following discussion, assume that the graph G is a tree. Define

$$Q_G(x) = \sum_{t=0}^{\infty} p_G(t)(1-x)^{t/2}. \tag{1.13}$$

The generating function variable x is defined in this way for notational convenience in later calculations. Note that since the graph G is a tree only integer exponents appear on $1-x$. Let $p_G^1(t)$ be the probability that a random walk which leaves the root at time zero and walks t steps, returns to the root for the first time. Define the generating function

$$P_G(x) = \sum_{t=0}^{\infty} p_G^1(t)(1-x)^{t/2}. \tag{1.14}$$

By decomposing a walk which returns to the root into the first return walk, the second return walk etc. we find the relation

$$Q_G(x) = \sum_{n=0}^{\infty} (P_G(x))^n = \frac{1}{1-P_G(x)}. \tag{1.15}$$

Let n be the smallest nonnegative integer for which $Q_\tau^{(n)}(x)$, the n–th derivative of $Q(x)$, diverges as $x \longrightarrow 0$. If

$$(-1)^n Q_\tau^{(n)}(x) \asymp x^{-\alpha} \tag{1.16}$$

for some $\alpha \in [0,1)$ then clearly

$$d_s = 2(1-\alpha+n), \tag{1.17}$$

if d_s exists. For random graphs, the same relation holds between the singular behaviour of $\langle Q_\tau^{(n)}(x) \rangle_\nu$ as $x \longrightarrow 0$ and the annealed spectral dimension. All statements about the spectral dimension of graphs in the following chapters are made under the assumption that it exists.

1.3.4 Hausdorff dimension

Another notion of dimension for graphs is the *Hausdorff dimension* which is defined in terms of how the volume of a ball scales with its radius. For a graph $G = (V, E)$, denote by $B_R(G)$ the subgraph of G which has a vertex set

$$V_R(G) = \{v \in V \mid d_G(v, r) \leq R\} \tag{1.18}$$

and an edge set

$$\{\{v, w\} \in E \mid v, w \in V_R(G), v \neq w\}. \tag{1.19}$$

We call $B_R(G)$ the ball of radius R centered on the root r. The Hausdorff dimension of the graph G is defined as d_H provided that

$$|B_R(G)| \asymp R^{d_H}. \tag{1.20}$$

As for the spectral dimension, this definition only makes sense on an infinite graph. On a connected, locally finite graph, d_H is independent of the choice of a root. On the hyper–cubic lattice $\mathbb{Z}^d$ it holds that $d_H = d$ but in general d_H is not an integer.

The Hausdorff dimension can also be defined in different ways for random graphs. First of all the graphs might have, ν almost surely, a Hausdorff dimension d_H as defined above and secondly we define the *annealed Hausdorff dimension* as $\bar{d}_H$ provided that

$$\langle |B_R(G)| \rangle_\nu \asymp R^{\bar{d}_H}. \tag{1.21}$$

The Hausdorff dimension and the spectral dimension do not necessarily agree, but under certain conditions the inequality

$$\frac{2d_H}{1 + d_H} \leq d_s \leq d_H \tag{1.22}$$

holds [30]. For trees which satisfy certain regularity conditions, the left inequality is saturated [13], i.e.

$$d_s = \frac{2d_H}{1 + d_H}. \tag{1.23}$$

Examples of random trees which satisfy (1.23) are the uniform spanning tree on $\mathbb{Z}^2$ ($d_H = 8/5$ and $d_s = 16/13$) [14] and generic trees ($d_H = 2$ and $d_s = 4/3$) [38]. Examples of random trees which satisfy (1.22) but not necessarily (1.23) are the random combs studied in [37].

It is beyond the scope of this thesis to discuss the relations (1.22) and (1.23) in detail. However, we will see that most of the random tree models we study have Hausdorff and spectral dimensions which satisfy (1.23). An exception to this is the condensed phase of the equilibrium statistical mechanical models.

There is another definition of the Hausdorff dimension which applies when one considers finite, randomly growing graphs. Let ν_N, $N = 1, 2, \ldots$ be probability distributions on a set of graphs $\mathcal{G}$, concentrated on graphs of volume N, and defined from ν_{N-1} by a growth rule. The Hausdorff dimension is usually defined in terms of how the average value of some typical distance in the graph (the maximum distance between vertices, the mean distance of vertices from the root, etc.) scales in relation to the volume of the graph as it grows. More precisely, we define the radius of the graph G by

$$R_G = \frac{1}{2|G|} \sum_{v} d_G(r, v)\, \sigma(v). \tag{1.24}$$

Then we define the Hausdorff dimension as d_H if

$$\langle R_G \rangle_{\nu_N, G \in \mathcal{G}} \sim N^{1/d_H} \tag{1.25}$$

as $N \longrightarrow \infty$. By $f(x) \sim g(x)$ as $x \longrightarrow \infty$ we mean that the limit of the ratio of $f(x)$ and $g(x)$ is a positive constant. This definition of d_H should be independent of the choice of a root r. If the measures ν_N converge to a measure ν concentrated on infinite graphs, the definition is expected to coincide with the previous one in (1.21) provided that ν is concentrated on sufficiently regular graphs. We will see examples of random tree models where this is the case.

Part I

Condensation in tree models

2

Caterpillars with a local action

In this chapter we consider an equilibrium statistical mechanical model of a certain class of trees which are referred to as *caterpillars*. We start by defining the set of caterpillars of a finite volume and introduce a Gibbs measure on this set, which is constructed from a local action. We analyse the model by the use of generating functions and show that there exist two phases, an *elongated phase* and a *condensed phase*. The asymptotic behaviour of the finite volume partition function is established in both phases and at the critical line separating the phases. We prove convergence of the finite volume measures to a measure on the set of infinite caterpillars and characterize it. The Hausdorff and spectral dimensions are then calculated, with respect to the infinite volume measure, in both phases and on the critical line. We conclude by briefly discussing a model which generalizes the caterpillar model and brings us one step closer to the model of branched polymers which is the subject of the next chapter.

2.1 Caterpillars

A finite caterpillar is a finite graph which consists of a linear graph, which we call the *spine*, to which leaves (i.e. individual links) are attached. We mark the end vertices of the linear graph by r_1 and r_2 and call r_1 the *root* of the caterpillar. Both these

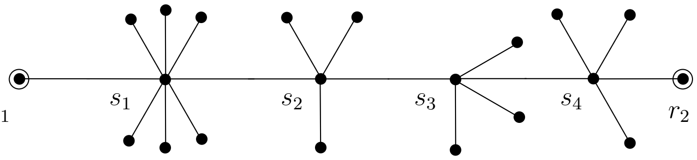

Figure 2.1: An example of a finite caterpillar graph.

vertices have order one by definition. Furthermore, we will view the caterpillars as planar graphs so we distinguish between left leaves and right leaves, see Fig. 2.1. The assumption of planarity is not essential. We denote the set of all caterpillars with N edges by C_N. For a caterpillar $\tau \in C_N$, denote the graph distance between r_1 and r_2 by $\ell(\tau)$ and call it the *length* of the caterpillar. For a caterpillar of length ℓ we denote the vertices on the spine between r_1 and r_2 by $s_1, \ldots, s_{\ell-1}$.

Let w_n, $n = 1, 2, \ldots$, be a sequence of nonnegative numbers which will be called *weight factors* or *branching weights*. The weight of a caterpillar $\tau \in C_N$ is defined as

$$w(\tau) = \prod_{i \in V(\tau) \setminus \{r_1, r_2\}} w_{\sigma(i)}. \tag{2.1}$$

We define the finite volume partition function by

$$Z_N = \sum_{\tau \in C_N} w(\tau) \tag{2.2}$$

and a probability distribution on C_N by

$$\nu_N(\tau) = \frac{w(\tau)}{Z_N}. \tag{2.3}$$

The weight factors w_n, or alternatively the measures ν_N, define what we call a *caterpillar ensemble*. The object of this chapter is to study the caterpillar ensemble for different classes of weights w_n and give a complete and rigorous categorization of different phases of the model.

Since the probability of a given caterpillar only depends on the order of its vertices, an equivalent way of defining this ensemble is the following. If $\tau \in C_N$ consider the finite sequence $c(\tau) = (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_{\ell-1}))$ and assign to it the probability

$$\tilde{\nu}_N(c(\tau)) = \nu_N(\tau) \prod_{i=1}^{\ell(\tau)-1} (\sigma(s_i) - 1). \tag{2.4}$$

The product factor in (2.4) accounts for the number of different caterpillars which correspond to the same sequence $c(\tau)$. Define the set $\tilde{C}_N = \{c(\tau) \mid \tau \in C_N\}$. It is clear that (C_N, ν_N) is equivalent to $(\tilde{C}_N, \tilde{\nu}_N)$ in the sense that $\nu_N(\tau)$ only depends on $c(\tau)$. This allows us to extend the notion of finite caterpillars to infinite ones:

$$\tilde{C} = \left\{ (b_i)_{i=1}^{k-2} \;\middle|\; k, b_i \in \{2, 3, \ldots\} \cup \{\infty\} \right\} \tag{2.5}$$

where $k = 2$ corresponds to the unique caterpillar of length $\ell = 1$. Note that an element in $\tilde{C}$ which has infinite terms and/or infinite length has no counterpart in C_N for any N. We denote the subset of finite caterpillars in $\tilde{C}$ by $\tilde{C}'$.

In the following sections we study the limit of the measures $\tilde{\nu}_N$ as $N \longrightarrow \infty$. In order to deal properly with convergence questions we need to define a topology on $\tilde{C}$. For a caterpillar $a = (a_1, a_2, \ldots) \in \tilde{C}$ we define the sequence

$$\tilde{B}_R(a) = (\min\{a_1, R\}, \min\{a_2, R\}, \ldots, \min\{a_{\min\{\ell(a)-1, R\}}, R\}). \tag{2.6}$$

We then define a metric $\tilde{d}$ on $\tilde{C}$ by

$$\tilde{d}(a, b) = \inf \left\{ \frac{1}{R} \;\middle|\; \tilde{B}_R(a) = \tilde{B}_R(b) \right\} \tag{2.7}$$

for any $a, b \in \tilde{C}$. It is straightforward to show that this definition satisfies the axioms for a metric. We now state and prove a few properties of the metric space $(\tilde{C}, \tilde{d})$.

Proposition 2.1.1 *The metric space* $(\tilde{C}, \tilde{d})$ *is compact.*

Proof Take a sequence $(c_n)_{n\in\mathbb{N}}$ in $\tilde{C}$. Note that for every R the set $\{\tilde{B}_R(b) \mid b \in \tilde{C}\}$ is finite. Therefore there exists a subsequence $(c_{n_i})_{i\in\mathbb{N}}$ such that $\tilde{B}_R(c_{n_i})$ is constant in i and it can be chosen such that $\tilde{B}_i(c_{n_j}) = \tilde{B}_i(c_{n_i})$ for all $i \leq j$. Thus, there is a unique caterpillar $c \in \tilde{C}$ such that $\tilde{B}_i(c) = \tilde{B}_i(c_{n_i})$ for all i and $c_{n_i} \longrightarrow c$ as $i \longrightarrow \infty$.

□

Denote the open ball in $\tilde{C}$ centered on c_0 and with radius r by

$$\mathcal{B}_r(c_0) = \{c \in \tilde{C} \mid \tilde{d}(c_0, c) < r\}. \tag{2.8}$$

Proposition 2.1.2 *For* $r > 0$ *and* $c_0 \in \tilde{C}$*, the ball* $\mathcal{B}_r(c_0)$ *is both open and closed. Moreover, if* $c_1 \in \mathcal{B}_r(c_0)$ *then* $\mathcal{B}_r(c_1) = \mathcal{B}_r(c_0)$.

Proof It is easy to see that open balls are closed since the positive values of $\tilde{d}$ form a discrete set but the parameter r is continuous. To prove the second statement choose a $c_1 \in \mathcal{B}_r(c_0)$ and a $c_2 \in \mathcal{B}_r(c_1)$. Clearly, $\tilde{B}_R(c_1) = \tilde{B}_R(c_0)$ and $\tilde{B}_R(c_1) = \tilde{B}_R(c_2)$ for all $R < 1/r$ so $\tilde{B}_R(c_0) = \tilde{B}_R(c_2)$ for all $R < 1/r$. Therefore

$$\tilde{d}(c_2, c_0) \leq \inf \left\{ \frac{1}{R} \;\middle|\; \tilde{B}_R(c_2) = \tilde{B}_R(c_0), R < 1/r + 1 \right\} < r \tag{2.9}$$

and thus $c_2 \in \mathcal{B}_r(c_0)$ which shows that $\mathcal{B}_r(c_1) \subseteq \mathcal{B}_r(c_0)$. With the same argument one shows that $\mathcal{B}_r(c_0) \subseteq \mathcal{B}_r(c_1)$ and therefore the equality is established.

□

Proposition 2.1.3 *The set $\tilde{C}'$ of finite caterpillars is a countable dense subset of $\tilde{C}$.*

Proof The set $\tilde{C}'$ is clearly countable since it is a countable union of finite sets. To prove that it is dense in $\tilde{C}$ take a $c \in \tilde{C}$. The sequence $\left(\tilde{B}_n(c)\right)_{n\in\mathbb{N}}$ is in $\tilde{C}'$ and converges to c.

□

2.2 Critical point and the different phases

Define the finite volume partition function with fixed distance ℓ between r_1 and r_2 as

$$Z_{N,\ell} = \sum_{\tau \in C_N, \ell(\tau)=\ell} w(\tau). \tag{2.10}$$

It is useful to work with the generating functions

$$\mathcal{Z}(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N \tag{2.11}$$

and

$$g(z) = \sum_{n=0}^{\infty} w_{n+1} z^n \tag{2.12}$$

with radii of convergence ζ_0 and ρ, respectively, both of which we assume to be nonzero. Define also

$$\mathcal{Z}_\ell(\zeta) = \sum_{N=1}^{\infty} Z_{N,\ell} \zeta^N. \tag{2.13}$$

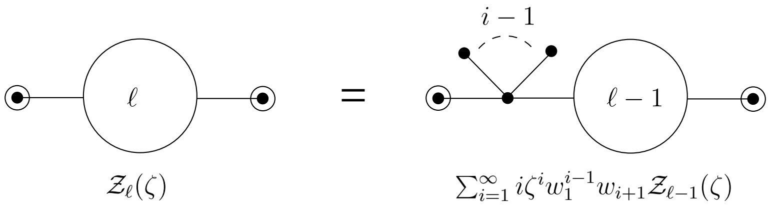

Figure 2.2: An illustration of the recursion (2.15).

Then it is clear that

$$\mathcal{Z}(\zeta) = \sum_{\ell=1}^{\infty} \mathcal{Z}_\ell(\zeta). \tag{2.14}$$

We have the recursion relation

$$\mathcal{Z}_\ell(\zeta) = \zeta g'(w_1\zeta)\mathcal{Z}_{\ell-1}(\zeta), \tag{2.15}$$

for any $\ell \geq 2$, see Fig. 2.2. Using the above equation and $\mathcal{Z}_1(\zeta) = \zeta$ gives

$$\mathcal{Z}_\ell(\zeta) = \zeta\Big(\zeta g'(w_1\zeta)\Big)^{\ell-1} \tag{2.16}$$

and by (2.14)

$$\mathcal{Z}(\zeta) = \frac{\zeta}{1-\zeta g'(w_1\zeta)}. \tag{2.17}$$

From (2.17) we see that ζ_0 is the smallest solution of the equation

$$\zeta g'(w_1\zeta) = 1 \tag{2.18}$$

on the interval $(0, \rho/w_1)$ if such a solution exists. If it does not exist then $\zeta_0 = \rho/w_1$.

If $\zeta_0 < \rho/w_1$ then g is analytic at $w_1\zeta_0$ and we say that we have a *generic ensemble*. This has been called the elongated or fluid phase by other authors [20]. If $\zeta_0 = \rho/w_1$ we have a *nongeneric ensemble*. Notice that if $\rho = \infty$ then the ensemble is always generic. For nongeneric ensembles we therefore have finite ρ. In that case we can always choose $\rho = 1$ by scaling the weights $w_n \to w_n\rho^{n-1}$. This scaling does not affect the probabilities (2.3).

Now consider weight factors with $\rho = 1$ and let w_1 be a free parameter. The genericity condition is then $\frac{1}{w_1}g'(1) > 1$, i.e. $w_1 < w_c$ where

$$w_c \equiv g'(1) = \sum_{n=2}^{\infty}(n-1)w_n \tag{2.19}$$

is a critical value for w_1. If $w_1 = w_c$ we have a nongeneric ensemble which we refer to as *critical* and if $w_1 > w_c$ we have a nongeneric ensemble which we refer to as *subcritical*. This phase has been called the condensed phase in the literature [20]. In the following subsection we determine the asymptotic behavior as $N \longrightarrow \infty$ of the finite volume partition functions Z_N for the different phases.

2.2.1 The generic phase

Let w_n be weight factors with $w_1 \neq 0$ and $w_n \neq 0$ for some $n > 2$ which lead to a generic ensemble.

Lemma 2.2.1 *Under the stated assumptions on the weight factors, the asymptotic behaviour of* Z_N *is given by*

$$Z_N = \frac{1}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)}\zeta_0^{-N}(1 + O(N^{-1})) \tag{2.20}$$

if the integers $n > 0$ *for which* $w_{n+1} \neq 0$ *have no common divisors greater than 1. Otherwise, if their greatest common divisor is* $d \geq 2$*, then*

$$Z_N = \frac{d}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)}\zeta_0^{-N}(1 + O(N^{-1})) \tag{2.21}$$

if $N = 1 \bmod d$*, and* $Z_N = 0$ *otherwise.*

The proof of this Lemma is standard, cf. [42], where the corresponding result for generic trees is established. We include it here for completeness.

Proof First assume that $\gcd\{n|n > 0, w_{n+1} \neq 0\} = 1$. The function

$$f(\zeta) = 1 - \zeta g'(w_1\zeta) \tag{2.22}$$

has a zero at ζ_0. The multiplicity of the zero is 1 since it is easily seen that $f'(\zeta_0) \neq 0$. We therefore see that $\mathcal{Z}$ has a simple pole at ζ_0 and since $\gcd\{n|n > 0, w_{n+1} \neq 0\} = 1$ there is no other pole on the circle $|\zeta| = \zeta_0$. By Taylor expanding $g'(w_1\zeta)$ around

$w_1\zeta_0$ we get

$$\mathcal{Z}(\zeta) = \frac{1}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)} \left(\frac{\zeta}{\zeta_0 - \zeta}\right) + O(\zeta_0 - \zeta). \tag{2.23}$$

Now define

$$h(\zeta) = Z(\zeta) - \frac{1}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)} \left(\frac{\zeta}{\zeta_0 - \zeta}\right) \tag{2.24}$$

and denote its radius of convergence by R. The function h has no poles for $|\zeta| \leq \zeta_0$ and therefore $R > \zeta_0$. We conclude that for any $\epsilon > 0$ the coefficients of h cannot grow faster than

$$[\zeta^n]\{h(\zeta)\} = O\left(\frac{1}{R} + \epsilon\right)^n \tag{2.25}$$

for n large. Therefore,

$$Z_N = \frac{1}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)} [\zeta^N] \left(\frac{\zeta}{\zeta_0 - \zeta}\right) + O\left(\frac{1}{R} + \epsilon\right)^N . \tag{2.26}$$

The result follows by straightforward calculation of $[\zeta^N]\left(\frac{\zeta}{\zeta_0 - \zeta}\right)$ and noticing that

$$O(\zeta_0^{-N}/N) > O(1/R + \epsilon)^N \tag{2.27}$$

for ϵ small enough.

Now assume that $d = \gcd\{n | n > 1, w_{n+1} \neq 0\} \geq 2$. Then the function g is of the form $g(z) = \tilde{g}(z^d)$ and therefore $\mathcal{Z}$ has d simple poles $\zeta_0, \zeta_1, \ldots, \zeta_{d-1}$ on the circle $|\zeta| = \zeta_0$, which are the d'th roots of ζ_0^d. We then define

$$h(\zeta) = \mathcal{Z}(\zeta) - \sum_{i=0}^{d-1} \frac{1}{g'(w_1\zeta_i) + \zeta_i w_1 g''(w_1\zeta_i)} \left(\frac{\zeta}{\zeta_i - \zeta}\right) \tag{2.28}$$

which is analytic with radius of convergence $R > \zeta_0$. We then get with the same argument as above that

$$\begin{aligned} Z_N &= \sum_{i=0}^{d-1} \frac{1}{g'(w_1\zeta_i) + \zeta_i w_1 g''(w_1\zeta_i)} \zeta_i^{-N}(1 + O(N^{-1})) \\ &= \frac{1}{\zeta_0 g'(w_1\zeta_0) + \zeta_0^2 w_1 g''(w_1\zeta_0)} \sum_{i=0}^{d-1} \zeta_i^{-N+1}(1 + O(N^{-1})) \end{aligned} \tag{2.29}$$

where the latter equality follows from $\zeta_i g'(w_1\zeta_i) = \zeta_0 g'(w_1\zeta_0)$ and

$\zeta_i^2 g''(w_1\zeta_i) = \zeta_0^2 g''(w_1\zeta_0)$ for $i = 0, \dots, d-1$. The last sum equals $d\zeta_0^{-N+1}$ if $N-1 = 0$ mod d but is zero otherwise. This completes the proof.

☐

2.2.2 The subcritical phase

We take $\rho = 1$ and $w_1 > w_c$ so that we are in the subcritical phase. We study a concrete model of weights $w_i,\ i \geq 2$ where

$$w_i = i^{-\beta}(1 + o(1)) \tag{2.30}$$

and let w_1 be a free parameter in the specified range. Figure 2.3 shows the phase diagram of the caterpillars. A necessary condition for being in the subcritical phase is $\beta > 2$ since otherwise $w_c = \infty$.

Lemma 2.2.2 *For the weights given in (2.30) and $w_1 > w_c$ we have*

$$Z_N = \frac{1}{(w_1 - w_c)^2} N^{1-\beta} w_1^N \big(1 + o(1)\big) \tag{2.31}$$

as $N \longrightarrow \infty$.

Proof We can write

$$Z_N = \sum_{\ell=1}^{N} Z_{N,\ell}. \tag{2.32}$$

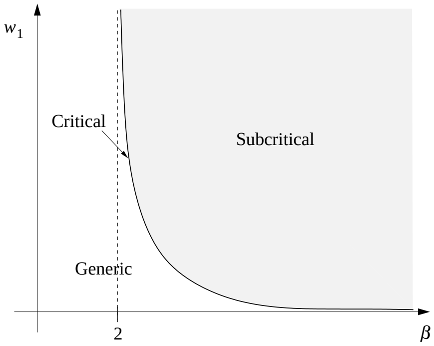

Figure 2.3: A diagram showing the different phases of the caterpillars.

Define a sequence of functions f_N on the positive integers by

$$f_N(\ell) = w_1^{-N} N^{\beta-1} Z_{N,\ell} = w_1^{-\ell} N^{\beta-1} \sum_{N_1+\ldots+N_{\ell-1}=N-\ell} \prod_{i=1}^{\ell-1} (N_i+1) w_{N_i+2}. \tag{2.33}$$

We begin by showing that

$$\lim_{N\longrightarrow\infty} f_N(\ell) = \frac{1}{w_c^2}(\ell-1)\left(\frac{w_c}{w_1}\right)^{\ell} \equiv f(\ell). \tag{2.34}$$

There is at least one index i in the sum defining $f_N(\ell)$ such that $N_i \geq \frac{N-\ell}{\ell-1}$. If there is another index $j \neq i$ such that $N_j > A$ where $A > 1$ is a constant then we get an upper bound on that contribution to $f_N(\ell)$ of the form

$$\begin{aligned}
& w_1^{-\ell} w_N^{-1} N^{-1} (\ell-1)^2 \sum_{\substack{N_1+\ldots+N_{l-1}=N-\ell \\ N_1 \geq \frac{N-\ell}{\ell-1} \\ N_2 > A}} \frac{N_1+1}{(N_1+2)^{\beta}} \prod_{i=2}^{\ell-1} (N_i+1) w_{N_i+2} \\
\leq\ & C(\ell) \frac{N^{\beta}}{(N+\ell-2)^{\beta}} \sum_{N_3,\ldots,N_{\ell-1}\geq 0} \sum_{N_2>A} (N_2+1) w_{N_2+2} \prod_{i=3}^{\ell-1} (N_i+1) w_{N_i+2} \\
\leq\ & D(\ell) w_c^{\ell-3} \sum_{N_2>A} (N_2+1) w_{N_2+2}
\end{aligned} \tag{2.35}$$

where $C(\ell)$ and $D(\ell)$ are numbers which only depend on ℓ. The last expression goes to zero as $A \longrightarrow \infty$ since $g'(1)$ is finite. The remaining contribution to $f_N(\ell)$ is

$$\begin{aligned}
& w_1^{-\ell} w_N^{-1} N^{-1} \sum_{k=1}^{\ell-1} \sum_{\substack{N_1+\ldots N_{\ell-1}=N-\ell \\ N_k \geq \frac{N-\ell}{\ell-1} \\ N_j \leq A, \quad j\neq k}} \prod_{i=1}^{\ell-1} (N_i+1) w_{N_i+2} \\
\xrightarrow[N\to\infty]{}\ & w_1^{-\ell} (\ell-1) \left(\sum_{n=0}^{A} (n+1) w_{n+2} \right)^{\ell-2} \\
\xrightarrow[A\to\infty]{}\ & w_c^{-2} (\ell-1) \left(\frac{w_c}{w_1}\right)^{\ell}
\end{aligned}$$

which proves (2.34).

Note that $f_N(\ell) = 0$ if $\ell > N$ and therefore it is clear that $f_N(\ell)$ is summable for every N. We also see that $f(\ell)$ is summable since $w_1 > w_c$. For $\ell \leq N$

$$\begin{aligned} f_N(\ell) &= w_1^{-\ell} N^{\beta-1} \sum_{N_1+\ldots+N_{\ell-1}=N-\ell} \prod_{i=1}^{\ell-1} (N_i+1) w_{N_i+2} \\ &\leq C w_1^{-\ell} N^{\beta-1} (\ell-1) \sum_{\substack{N_1+\ldots+N_{\ell-1}=N-\ell \\ N_1 \geq \frac{N-\ell}{\ell-1}}} \frac{N_1+1}{(N_1+2)^\beta} \prod_{i=2}^{\ell-1} (N_i+1) w_{N_i+2} \\ &\leq C \frac{1}{w_c^2} \left(\frac{w_c}{w_1}\right)^\ell \frac{N^{\beta-1}(N-1)}{\left(\frac{N-\ell}{\ell-1}+2\right)^\beta} \leq C'(\ell-1)^\beta \left(\frac{w_c}{w_1}\right)^\ell \end{aligned} \tag{2.36}$$

where C and C' are positive constants. The first inequality in (2.36) is obtained by observing that at least one of the indices N_i must be larger than $\frac{N-\ell}{\ell-1}$ and in the second one we used the definition of w_c. It follows that the sequence $(f_N)_{N=1}^\infty$ is dominated by a summable function and we can calculate the limit

$$\lim_{N\to\infty} \left(w_1^{-N} w_N^{-1} N^{-1} Z_N\right) = \lim_{N\to\infty} \sum_{\ell=1}^\infty f_N(\ell) = \sum_{\ell=1}^\infty f(\ell) = \frac{1}{(w_1-w_c)^2}. \tag{2.37}$$

This completes the proof.

□

From the above lemma we obtain the following result

$$\lim_{N\to\infty} \frac{Z_{N,\ell}}{Z_N} = (\ell-1)\left(1-\frac{w_1}{w_c}\right)^2 \left(\frac{w_c}{w_1}\right)^\ell \tag{2.38}$$

which indicates that with probability 1 the caterpillar has finite length which is exponentially distributed with a parameter w_c/w_1. If the length of an infinite caterpillar is $\ell < \infty$ it is clear that it has one or more vertices of infinite order. The inequality (2.35) shows that there can be at most one vertex of infinite order in the limit $N \longrightarrow \infty$. We will state this observation more precisely in the next section when we prove the convergence of the measures $\tilde{\nu}_N$.

2.2.3 The critical line

We take $\rho = 1$ and $w_1 = w_c$ so that we are on the critical line and choose the weights as in (2.30) where $\beta > 2$. We make the extra assumption that the generating function

$\mathcal{Z}$ is analytic in a domain

$$D(\Delta,\phi,\zeta_0) = \{\zeta \;:\; |\zeta| < \zeta_0 + \Delta, \phi/2 < \mathrm{Arg}(\zeta - \zeta_0) < 2\pi - \phi/2, \zeta \neq \zeta_0\} \tag{2.39}$$

for some $\Delta > 0$ and some angle $\phi \in (0, \pi/2)$, and that

$$\mathcal{Z}(\zeta) \sim (\zeta_0 - \zeta)^{2-\beta} \tag{2.40}$$

as $\zeta \longrightarrow \zeta_0$ on $D(\Delta,\phi,\zeta_0)$. This condition allows one to deduce the asymptotic behaviour of Z_N, the coefficients of $\mathcal{Z}$, see [42, Section VI. 3 pages 389-392] for a detailed explanation. The above condition on the weights is not empty. For example, the explicit choice

$$w_1 = 1, \quad w_2 = 0 \quad \text{and} \quad w_n = \frac{(\beta-2)\Gamma_{\beta-2}(n-2)}{(n-1)!}, \; n \geq 3 \tag{2.41}$$

where $\Gamma_\gamma(n) = (n-1-\gamma)(n-2-\gamma)\cdots(2-\gamma)(1-\gamma)$, $n \geq 2$ and $\Gamma_\gamma(1) = 1$ yields $w_n \sim n^{-\beta}$ and

$$\mathcal{Z}(\zeta) = \frac{\zeta}{1 - \zeta(1-(1-\zeta)^{\beta-2})}, \tag{2.42}$$

which satisfies the above conditions for $\zeta_0 = 1$, any $\Delta > 0$ and any $\phi \in (0, \pi/2)$. We will encounter this particular choice of weights again at the end of Part II. A straightforward application of [42, Corollary VI. 1] gives the following lemma

Lemma 2.2.3 *Choose branching weights as in (2.30) with* $w_1 = w_c$*, such that* $\mathcal{Z}$ *is analytic in a domain* $D(\Delta,\phi,\zeta_0)$ *for some* $\Delta > 0$*,*$\phi \in (0, \pi/2)$ *and obeys*

$$\mathcal{Z}(\zeta) \sim (\zeta_0 - \zeta)^{2-\beta} \tag{2.43}$$

as $\zeta \longrightarrow \zeta_0$ *on* $D(\Delta,\phi,\zeta_0)$*. Then*

$$Z_N \sim N^{\beta-3}\zeta_0^{-N}. \tag{2.44}$$

as $N \longrightarrow \infty$*.*

2.3 Convergence of the finite volume measures

In this section we prove that the measures $\tilde{\nu}_N$ converge weakly as $N \longrightarrow \infty$ to a measure $\tilde{\nu}$ and we give a complete description of $\tilde{\nu}$ for different phases of the model.

Theorem 2.3.1 *For the generic and nongeneric branching weights in Lemmas 2.2.1–2.2.3, the measures* $\tilde{\nu}_N$ *converge weakly as* $N \longrightarrow \infty$ *to a probability measure* $\tilde{\nu}$ *concentrated on the set of infinite caterpillars. If* $w_1 \leq w_c$, $\tilde{\nu}$ *is concentrated on the set of caterpillars of infinite length and the degrees of the vertices* $s_1, s_2, \ldots$ *on the spine are independently distributed by*

$$\phi(n) = \zeta_0 (n-1) w_n (w_1 \zeta_0)^{n-2}, \qquad n \geq 2. \tag{2.45}$$

If $w_1 > w_c$, $\tilde{\nu}$ *is concentrated on the set of caterpillars of finite length with exactly one vertex of infinite degree. The length of the spine is distributed by*

$$\psi(\ell) = (\ell - 1) \left(1 - \frac{w_1}{w_c}\right)^2 \left(\frac{w_c}{w_1}\right)^\ell. \tag{2.46}$$

All the vertices between r_1 *and* r_2 *are equally likely to be of infinite degree and the degree of the others are independently distributed by* [1]

$$\phi(n) = \frac{1}{w_c}(n-1) w_n. \tag{2.47}$$

Proof We define a family of sets

$$\mathcal{U} = \{\mathcal{B}_{\frac{1}{k}}(c) \mid k \in \mathbb{N},\ c \in \tilde{C}'\}. \tag{2.48}$$

From the properties of the metric space $(\tilde{C}, \tilde{d})$ the family $\mathcal{U}$ clearly satisfies (i) and (ii) in Section 1.3.2 and since $(\tilde{C}, \tilde{d})$ is compact, tightness is automatically fulfilled. It therefore only remains to prove property (iii). Choose a $k \in \mathbb{N}$ and a $c = (c_1, c_2, \ldots, c_{\ell(c)-1}) \in \tilde{C}'$ and define $A = \mathcal{B}_{\frac{1}{k}}(c)$. Denote the set of indices $i \leq \min\{k, \ell(c) - 1\}$ for which $c_i < k$ by $\underline{I}$ and the set of indices $i \leq \min\{k, \ell(c) - 1\}$ such that $c_i \geq k$ by $\overline{I}$. We consider seperately the following cases.

$\underline{w_1 < w_c}$: In this case we are in the generic phase so $w_1 \zeta_0 < \rho$ and $Z_N \sim \zeta_0^{-N}$ cf. Lemma 2.2.1. We assume that $\ell(c) \geq k$ and if this conditions is not fulfilled we get a simple special case of the calculations below. The set A is then given by

$$A = \{b \in \tilde{C} \mid b_i = c_i \text{ if } i \in \underline{I}, b_i \geq k \text{ if } i \in \overline{I},\ \ell(b) \geq k\}. \tag{2.49}$$

Denote the number of elements in $\overline{I}$ by K. Now, order the indices in $\overline{I}$ in increasing order and for a given caterpillar in A let N_i, $1 \leq i \leq K$ be the term in the caterpillar

---

[1] We use the same notation for the degree distribution as in the case $w_1 \leq w_c$.

corresponding to the i–th index in $\overline{I}$. We can then write

$$\tilde{\nu}_N(A) = Z_N^{-1} W_0 \sum_{\substack{N_1+\cdots+N_K+M=N+k-c_0 \\ N_j \geq k,\ \forall j}} Z_M \prod_{i=1}^{K} (N_i - 1) w_{N_i} w_1^{N_i-2} \tag{2.50}$$

where

$$W_0 = \prod_{i \in \underline{I}} (c_i - 1) w_{c_i} w_1^{c_i-2} \qquad \text{and} \qquad c_0 = \sum_{i \in \underline{I}} c_i. \tag{2.51}$$

First consider the contribution to the sum in (2.50) from terms for which $N_i \geq (N + k - c_0)/(K+1)$ for some $i = 1, \ldots, K$. It can be estimated from above by

$$C \sup\{(N_i - 1) w_{N_i} w_1^{N_i-2} \zeta_0^{N_i-2} \mid N_i \geq (N+k-c_0)/(K+1)\} \tag{2.52}$$

where C is a number independent of N. This clearly converges to zero as $N \longrightarrow \infty$ since $w_1 \zeta_0 < \rho$. The remaining contribution to the sum is from terms where $M \geq (N + k - c_0)/(K+1)$. We then find that

$$\tilde{\nu}_N(A) \longrightarrow \zeta_0^k \prod_{i \in \underline{I}} (c_i - 1) w_{c_i} (w_1 \zeta_0)^{c_i-2} \left(\sum_{i=k}^{\infty} (i-1) w_i (w_1 \zeta_0)^{i-2} \right)^K \tag{2.53}$$

as $N \longrightarrow \infty$. It is clear from the above calculations and the formula (2.53) that $\tilde{\nu}$ has the stated properties.

$\underline{w_1 = w_c}$: In this case $w_1 \zeta_0 = \rho = 1$ and $Z_N \sim N^{\beta-3} \zeta_0^{-N}$ where $\beta > 2$, cf. Lemma 2.2.3. We proceed as in the generic case up to Equation (2.52) which is replaced by the estimate

$$C \sup\{Z_N^{-1} \zeta_0^{-N} (N_i - 1) w_{N_i} \mid N_i \geq (N+k-c_0)/(K+1)\} \sim N^{2(2-\beta)} \tag{2.54}$$

which converges to zero as $N \longrightarrow \infty$ since $\beta > 2$. We then continue and get the result (2.53) as above.

$\underline{w_1 > w_c}$: In this case $w_1 \zeta_0 = 1$ and $Z_N \sim N^{-\beta+1} \zeta_0^{-N}$ cf. Lemma 2.2.2 where $\beta > 2$. First assume that $\ell(c) \geq k$ as in the previous cases. Then Equation (2.50) applies. However, the upper bound (2.52) no longer converges to zero. We therefore consider the contribution from terms for which two different numbers $n_1, n_2 \in \{N_1, \ldots, N_K, M\}$ obey $n_1 \geq (N + k - c_0)/(K+1)$ and $n_2 > J$ for some positive number $J > k$. As in (2.35), this contribution is estimated from above by

$$C \sum_{i>J} i^{-\beta+1} \tag{2.55}$$

where C is a positive number independent of N and J. This converges to zero as $J \longrightarrow \infty$ since $\beta > 2$. The only remaining contribution to $\tilde{\nu}_N(A)$ is

$$\begin{aligned} & Z_N^{-1} W_0 \Big(\sum_{i=1}^{K} \sum_{\substack{N_1+\cdots+N_K+M=N+k-c_0 \\ k \leq N_j \leq J,\ j \neq i,\ M \leq J \\ N_i \geq k}} Z_M \prod_{j=1}^{K} (N_j - 1) w_{N_j} w_1^{N_j-2} \\ & + \sum_{\substack{N_1+\cdots+N_K+M=N+k-c_0 \\ k \leq N_j \leq J,\ \forall j}} Z_M \prod_{j=1}^{K} (N_j - 1) w_{N_j} w_1^{N_j-2} \Big) \end{aligned}$$

$$\begin{aligned} \xrightarrow[N\to\infty]{} \quad & W_0 \zeta_0^{-k+2K+c_0} \left(K(w_1 - w_c)^2 \sum_{n=1}^{J} Z_n \zeta_0^{n} \Big(\sum_{n=k}^{J} (n-1) w_n \Big)^{K-1} \right. \\ & \left. + \Big(\sum_{n=k}^{J} (n-1) w_n \Big)^{K} \right) \\ \xrightarrow[J\to\infty]{} \quad & \zeta_0^{k} \prod_{i \in \underline{I}} (c_i - 1) w_{c_i} \Big(\sum_{n=k}^{\infty} (n-1) w_n \Big)^{K-1} \left(K(w_1 - w_c) + \sum_{n=k}^{\infty} (n-1) w_n \right). \end{aligned} \tag{2.56}$$

Now assume that $\ell(c) < k$. Then with precisely the same calculation (with no Z_M factor) one gets

$$\tilde{\nu}_N(A) \longrightarrow \zeta_0^{\ell(c)} \prod_{i \in \underline{I}} (c_i - 1) w_{c_i} \Big(\sum_{n=k}^{\infty} (n-1) w_n \Big)^{K-1} K(w_1 - w_c)^2 \tag{2.57}$$

as $N \longrightarrow \infty$. From (2.57) one sees that $\tilde{\nu}$ is concentrated on the set of caterpillars of finite length with the stated length distribution ψ. The estimate (2.55) shows that there appears precisely one vertex of infinite degree on the spine and one can deduce the distribution of the degree of the others from (2.57).

□

2.4 Dimensions of the different phases

The generic phase is very simple and it is one dimensional for any sensible notion of dimension. The same applies on the critical line when $g''(1) < \infty$. We state this in the following theorem.

Theorem 2.4.1 *For the generic branching weights in Lemma 2.2.1 and the critical branching weights in Lemma 2.2.3 with $\beta > 3$, it holds that*

$$\bar{d}_s = \bar{d}_H = 1 \tag{2.58}$$

and

$$d_s = d_H = 1 \tag{2.59}$$

almost surely.

Proof We start by considering the Hausdorff dimension. For an infinitely long random caterpillar $c \in (\tilde{C}, \tilde{\nu})$, let $(X_n(c))_n$ be a sequence of random variables corresponding to the number of leaves attached to the vertices $s_1, s_2, \ldots$ of c. Define $S_R(c) = \sum_{i=1}^{R} X_i(c)$. Then $|B_R(c)| = S_{R-1}(c) + R$. From (2.45) it is clear that

$$\langle |B_R| \rangle_{\tilde{\nu}} = (\zeta_0 g''(w_1\zeta_0) - 2)(R-1) + R. \tag{2.60}$$

Since $g''(w_1\zeta_0) < \infty$ it follows from (1.21) that the annealed Hausdorff dimension is $\bar{d}_H = 1$. By the strong law of large numbers

$$|B_R(c)|/R \longrightarrow \zeta_0 g''(w_1\zeta_0) - 1 < \infty \tag{2.61}$$

almost surely as $R \longrightarrow \infty$ which shows that $d_H = 1$ almost surely.

Next we find the spectral dimension by establishing bounds on the return probability generating function. Let c be an infinitely long caterpillar with the corresponding return and first return generating functions $Q_c(x)$ and $P_c(x)$. We get an upper bound on $Q_c(x)$ by throwing away all the leaves from the spine. Then, by the monotonicity results of [37] we find that

$$Q_c(x) \leq x^{-1/2} \tag{2.62}$$

which shows that $\bar{d}_s \geq 1$ and $d_s \geq 1$ almost surely. To get a lower bound on $Q_c(x)$ we use a slight modification of Lemma 7 in [38] which is the following. For all integers

$R > 1$ and $0 < x \leq 1$,

$$P_c(x) \geq 1 - \frac{1}{R-1} - x - x|B_R(c)|. \tag{2.63}$$

We then get, using (1.15),

$$Q_c(x) \geq \frac{1}{\frac{1}{R-1} + x + x|B_R(c)|} \tag{2.64}$$

and by Jensen's inequality

$$\langle Q_c(x)\rangle_{\tilde{\nu}} \geq \frac{1}{\frac{1}{R-1} + x + x\langle|B_R|\rangle_{\tilde{\nu}}}. \tag{2.65}$$

Choose $R = \left[x^{-1/2}\right]$. We find, using (2.60) and (2.65), that $\bar{d}_s \leq 1$. Using (2.64) and the fact that $|B_R(c)| \sim R$ almost surely shows that $d_s \leq 1$ almost surely.

□

Next we consider a point on the critical line where $g''(1) = \infty$. We see straight away from (2.60) that the annealed Hausdorff dimension is infinite in this case and Equation (2.65) provides no useful bound on the annealed spectral dimension. However we can obtain almost sure results on the dimensions.

Theorem 2.4.2 *For the critical branching weights in Lemma 2.2.3 with* $2 < \beta \leq 3$*, it holds that*

$$d_H = \frac{1}{\beta - 2} \tag{2.66}$$

almost surely.

Proof To make the notation more compact define $\gamma = \beta - 2$. We prove a stronger statement, namely that there exist constants C_1 and C_2 and for $\tilde{\nu}$–almost all caterpillars c a constant $R_c > 0$ such that

$$C_1(\log(R)^{-1}R)^{1/\gamma} \leq |B_R(c)| \leq (\lambda(R)R)^{1/\gamma} \tag{2.67}$$

for all $R \geq R_c$. Here, $\lambda(R)$ can by any positive function with the property that

$$\sum_{R=1}^{\infty} \frac{1}{R\lambda(R)} < \infty. \tag{2.68}$$

In particular we can choose $\lambda(R) = (\log(R))^\eta$ for any $\eta > 1$ which is sufficient to obtain the desired result. Let $(X_n(c))_n$ be the sequence of random variables defined in the proof of Theorem 2.4.1. Since $1/\gamma \geq 1$ it is clearly sufficient to prove the inequalities (2.67) for $S_R(c)$. Begin with the lower bound. Take $\kappa, \theta > 0$. Using Markov's inequality and the independence of the X_i's we get

$$\begin{aligned}
\mathbb{P}(S_R(c) < \kappa) &= \mathbb{P}\left(e^{-\theta S_R} > e^{-\theta\kappa}\right) \leq e^{\theta\kappa}\left(\mathbb{E}\left(e^{-\theta X_i(c)}\right)\right)^R \\
&= e^{\theta\kappa}\left(\zeta_0 g'(e^{-\theta})\right)^R.
\end{aligned}$$

Taylor expanding $g'(e^{-\theta})$ around 1 yields

$$g'(e^{-\theta}) = g'(1) - (1 - e^{-\theta})g''(\xi) \tag{2.69}$$

for some number $\xi \in (e^{-\theta}, 1)$. Since g'' is increasing, it holds that $g''(\xi) > g''(e^{-\theta})$. Estimating $g''(e^{-\theta})$ by an integral and using $\zeta_0 g'(1) = 1$ yields

$$\zeta_0 g'(e^{-\theta}) \leq 1 - C(1 - e^{-\theta})^\gamma \leq e^{-C(1-e^{-\theta})^\gamma} = e^{-C\theta^\gamma(1+O(\theta))} \tag{2.70}$$

where C is a constant. Now choose $\kappa = K(\log(R))^{-1/\gamma}R^{1/\gamma}$ and $\theta = 1/\kappa$. Then, for R large enough

$$\mathbb{P}(S_R(c) < K(\log(R))^{-1/\gamma}R^{1/\gamma}) \leq C_3 e^{-CK^{-\gamma}\log(R)} = C_3 R^{-CK^{-\gamma}} \tag{2.71}$$

where C_3 is a positive constant. Choosing $K = C_1$ small enough we see that

$$\sum_{R=1}^{\infty} \mathbb{P}(S_R(c) < C_1(\log(R))^{-1/\gamma}R^{1/\gamma}) < \infty \tag{2.72}$$

and therefore, by the Borel–Cantelli lemma, there exists a constant R_c such that $S_R(c) \geq C_1(\log(R))^{-1/\gamma}R^{1/\gamma}$ almost surely for all $R \geq R_c$.

The upper bound follows from [41, Theorem 2] which states, for our purposes, that the probability of the event

$$S_R(c) > a_R, \qquad \text{for infinitely many } R \tag{2.73}$$

is zero if the sum

$$\sum_{R=0}^{\infty} \mathbb{P}(X_k \geq a_R) \tag{2.74}$$

converges, where a_R is a positive sequence with the property that $a_R/R \longrightarrow \infty$ as $R \longrightarrow \infty$. Now

$$\mathbb{P}(X_k \geq a_R) \leq C_4 a_R^{-\gamma} \tag{2.75}$$

for a suitable constant C_4. Choosing $a_R = (\lambda(R)R)^{1/\gamma}$, where $\lambda(R)$ has the properties stated above, completes the proof.

□

Theorem 2.4.3 *For the critical branching weights in Lemma 2.2.3 with* $2 < \beta \leq 3$, *it holds that*

$$d_s = \frac{2}{\beta - 1} \tag{2.76}$$

almost surely.

Proof Let c be an infinitely long caterpillar. Equation (2.64) provides a lower bound on $Q_c(x)$ and Equation (6) in [38] provides an upper bound such that

$$\frac{1}{\frac{1}{R-1} + x + x|B_R(c)|} \leq Q_c(x) \leq R + \frac{2}{x|B_R(c)|}. \tag{2.77}$$

Using (2.67) for a suitable choice of $\lambda(R)$ we get $\tilde{\nu}$–almost surely the inequality

$$\frac{1}{\frac{1}{R-1} + x + x(\lambda(R)R)^{1/(\beta-2)}} \leq Q_c(x) \leq R + \frac{2}{xC_1(\log(R)^{-1}R)^{1/(\beta-2)}} \tag{2.78}$$

for all $R \geq R_c$ and R_c large enough. Choosing $R = [x^{-\frac{\beta-2}{\beta-1}}]$ we find that there are numbers $K_1(c)$ and $K_2(c)$ such that $\tilde{\nu}$–almost surely

$$K_1(c)\lambda([x^{-\frac{\beta-2}{\beta-1}}])^{-1}x^{-\frac{\beta-2}{\beta-1}} \leq Q_c(x) \leq K_2(c)\log([x^{-\frac{\beta-2}{\beta-1}}])x^{-\frac{\beta-2}{\beta-1}}. \tag{2.79}$$

This yields the desired result.

□

Theorem 2.3.1 implies that the Hausdorff dimension d_H of a random caterpillar in the subcritical phase is almost surely infinite since with probability one there is a ball of finite radius which contains infinitely many vertices. The annealed Hausdorff dimension is infinite by the same argument. Similarly, the spectral dimension is almost surely infinite because a random walk which hits the infinite order vertex returns to the root with probability 0. From the analysis below one can easily check that the return probability on a randomly chosen subcritical caterpillar τ, $p_\tau(t)$, decays faster than any power of t.

In the remainder of this section we show that, although the spectral dimension is almost surely infinite, the annealed spectral dimension is finite.

Theorem 2.4.4 *For subcritical caterpillars defined by the weight factors given in (2.30) with $w_1 > w_c$ it holds that*

$$\bar{d}_s = 2(\beta - 1). \tag{2.80}$$

Proof We will refer to the unique vertex of infinite order as the *trap*. If the walk hits the trap it returns to the root with probability zero. Therefore, the part of the caterpillar beyond the trap is irrelevant for the random walk. When finding the spectral dimension it is therefore natural to consider the probability that the trap is at a distance ℓ from the root instead of considering the probability of the total length of the caterpillar given in (2.46).

For a caterpillar of a given length, all the vertices between r_1 and r_2 are equally likely to be of infinite order so the probability that the trap is at a distance ℓ from root is given by

$$p(\ell) = \sum_{k=\ell+1}^{\infty} \frac{\psi(k)}{k-1} = \Big(1 - \frac{w_c}{w_1}\Big)\Big(\frac{w_c}{w_1}\Big)^{\ell-1}. \tag{2.81}$$

From now on we will disregard the part of the caterpillar beyond the trap. Let $B_{\ell,k}$ be the set of caterpillars with distance ℓ between root and trap and which have one vertex of order k and all other vertices of order no greater than k, with the exception of the trap of course. Let $a(k)$ be the probability that a given vertex on the spine between the root and the trap has order no greater than k. Then

$$a(k) = \sum_{q=2}^{k} \phi(q). \tag{2.82}$$

The probability that at least one of these vertices has order k and all the others have order no greater than k is then

$$c(k,\ell) = a(k)^{\ell-1} - a(k-1)^{\ell-1}. \tag{2.83}$$

The average return generating function for the subcritical caterpillars is then

$$Q(x) = \sum_{\ell=1}^{\infty} p(\ell) \sum_{k=2}^{\infty} c(k,\ell) \sum_{\tau \in B_{\ell,k}} \tilde{\nu}(\{c \mid c \in B_{\ell,k}\}) Q_\tau(x). \tag{2.84}$$

For a given distance ℓ between root and trap we denote by M_ℓ the linear subgraph which starts at the root and ends at the trap, see Fig. 2.4. The first return generating function for M_ℓ is given by

$$P_{M_\ell}(x) = 1 - \sqrt{x}\frac{(1+\sqrt{x})^\ell + (1-\sqrt{x})^\ell}{(1+\sqrt{x})^\ell - (1-\sqrt{x})^\ell}, \tag{2.85}$$

see e.g. [37]. Now attach $k-2$ links to each vertex of the graph M_ℓ except the root and the trap and denote the resulting graph by $M_{\ell,k}$, see Fig. 2.5. Then $M_{\ell,k}$ is the largest graph in the set $B_{\ell,k}$. Using the methods of [51] we find that the first return generating function for $M_{\ell,k}$ is

$$P_{M_{\ell,k}}(x) = \left(1 + \frac{k-2}{2}x\right) P_{M_\ell}(x_k(x)) \tag{2.86}$$

where

$$x_k(x) = \frac{\frac{(k-2)^2}{4}x^2 + (k-1)x}{\left(1+\frac{k-2}{2}x\right)^2}. \tag{2.87}$$

To find an upper bound on the spectral dimension of subcritical caterpillars we establish a lower bound on the n-th derivative of the average return generating function. Let n be the smallest positive integer such that $Q^{(n)}(x)$ diverges as $x \longrightarrow 0$. We see in the following calculations that we have to choose n such that $n+1 < \beta \leq n+2$. By (1.15) we find that $(-1)^n Q_\tau^{(n)} \geq (-1)^n P_\tau^{(n)}$ for any τ. Thus, by differentiating (2.84) n times and throwing away every term in the sum over ℓ except the $\ell = 2$ term,

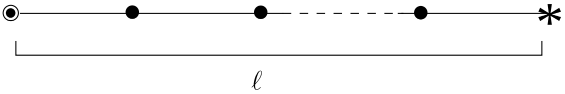

Figure 2.4: The graph M_ℓ. The root is denoted by a circled vertex and the trap by an asterisk.

we get the lower bound

$$(-1)^n Q^{(n)}(x) \geq (-1)^n \left(1 - \frac{w_c}{w_1}\right) \frac{w_c}{w_1} \sum_{k=2}^{\infty} \phi(k) P^{(n)}_{M_{2,k}}(x). \tag{2.88}$$

We easily find that

$$P_{M_{2,k}}(x) = \frac{1-x}{2+(k-2)x} \tag{2.89}$$

and show by induction that

$$P^{(n)}_{M_{2,k}}(x) = (-1)^n n! \frac{(k-2)^{n-1} k}{(2+(k-2)x)^{n+1}}. \tag{2.90}$$

Then, by (2.47) and (2.90),

$$\begin{aligned} (-1)^n \sum_{k=2}^{\infty} \phi(k) P^{(n)}_{M_{2,k}}(x) &= \frac{n!}{w_c} \sum_{k=2}^{\infty} \frac{(k-2)^{n-1} k^{1-\beta}(k-1)}{(2+(k-2)x)^{n+1}} \\ &\geq C x^{\beta-n-2} \int_x^{\infty} \frac{y^{n+1-\beta}}{(2+y)^{n+1}} dy \end{aligned} \tag{2.91}$$

where $C > 0$ is a constant. If $\beta < n+2$ the last integral is convergent when $x \longrightarrow 0$ but if $\beta = n+2$ it diverges logarithmically. In both cases we get an upper bound for the annealed spectral dimension $\bar{d}_s \leq 2(\beta - 1)$.

To find a lower bound on the spectral dimension of subcritical caterpillars we establish an upper bound on the n-th derivative of the average return generating function. First note that $1 > a(k) = a(k-1) + \phi(k)$ and therefore

$$\begin{aligned} c(k,\ell) &= (a(k) - a(k-1)) \\ &\times \left(a(k)^{\ell-2} + a(k)^{\ell-3} a(k-1) + \ldots + a(k) a(k-1)^{\ell-3} + a(k-1)^{\ell-2}\right) \\ &\leq \phi(k)(\ell - 1). \end{aligned} \tag{2.92}$$

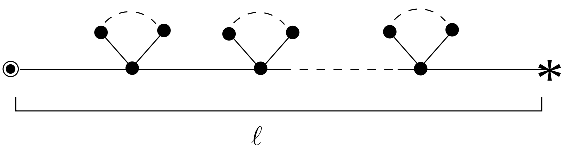

Figure 2.5: The graph $M_{\ell,k}$.

Now consider a caterpillar $\tau \in B_{\ell,k}$ and the graph M_ℓ. Denote the vertices on the spine of M_ℓ between the root and the trap by $s_1, s_2, \ldots, s_{\ell-1}$. One can obtain the graph τ from M_ℓ by attaching $m_\tau(s_i)$ links to s_i, $i = 1, \ldots, \ell - 1$ where $0 \leq m_\tau(s_i) \leq k - 2$. Using the methods of [51] we can write

$$Q_\tau(x) = \sum_{\substack{\omega:\ r_1 \longrightarrow r_1 \\ \text{on } M_\ell}} K_\tau(x, \omega) W_{M_\ell}(\omega)(1 - x)^{|\omega|/2} \tag{2.93}$$

where the sum is over all random walks ω on M_ℓ which begin and end at the root,

$$K_\tau(x, \omega) = \prod_{\substack{t=1 \\ \omega_t \in \{s_1, \ldots, s_{\ell-1}\}}}^{|\omega|-1} \left(1 + \frac{m_\tau(\omega_t)}{2} x\right)^{-1} \quad \text{and} \tag{2.94}$$

$$W_{M_\ell}(\omega) = \prod_{t=0}^{|\omega|-1} (\sigma(\omega_t))^{-1}. \tag{2.95}$$

The i–th derivative of the function $K_\tau(x, \omega)$ can be estimated as

$$(-1)^i \frac{d^i}{dx^i} K(x, \omega) \leq H(|\omega|) \frac{(k-2)^i}{(2 + (k-2)x)^i} \tag{2.96}$$

where H is a polynomial with positive coefficients. From the relation (1.15) and the explicit formula (2.85) one can easily see that $(-1)^i Q_{M_\ell}^{(i)}(0)$ is a positive polynomial in ℓ of degree $2i + 1$. Therefore, differentiating (2.93) n times and using the estimate (2.96) we get the upper bound

$$(-1)^n Q_\tau^{(n)}(x) \leq \sum_{i=0}^{n} S_i(\ell) \frac{(k-2)^i}{(2 + (k-2)x)^i} \tag{2.97}$$

where the S_i are positive polynomials in ℓ. Differentiating (2.84) n times w.r.t. x and using the estimates (2.92) and (2.97) we finally obtain

$$(-1)^n Q^{(n)}(x) \leq \sum_{i=0}^{n} \sum_{\ell=1}^{\infty} p(\ell) S_i(\ell)(\ell - 1) \sum_{k=2}^{\infty} \phi(k) \frac{(k-2)^i}{(2 + (k-2)x)^i}. \tag{2.98}$$

The sum over ℓ is convergent since S_i is a polynomial in ℓ and $p(\ell)$ decays exponentially. The sum over k is estimated from above by an integral as in (2.91) which yields a lower bound on the annealed spectral dimension $\bar{d}_s \geq 2(\beta - 1)$. This proves (2.80).

□

2.5 Generalization of the caterpillar model

The caterpillar model can be generalized to more complicated tree models by replacing the leaves on the spine by trees with vertices of orders bounded by K, the caterpillars correspond to $K = 1$. We will not go into details of the calculations for this model, however using a similar analysis as for the caterpillars, one obtains two phases: a fluid phase (generic) and a condensed phase (nongeneric), separated by a critical value of w_1 given by

$$w_c(K) = g'(1) - \sum_{n=2}^{K} w_n. \tag{2.99}$$

In the fluid phase, the finite volume probability measures converge to a measure concentrated on trees with an infinite spine with critical Galton–Watson[2] outgrowths analogous to the generic trees in [38]. In the condensed phase the measures converge to trees with spine of a finite length ℓ distributed by

$$\psi(\ell, K) = (\ell - 1)\left(1 - \frac{w_1}{w_c(K)}\right)^2 \left(\frac{w_c(K)}{w_1}\right)^\ell. \tag{2.100}$$

Exactly one of the vertices on the spine has infinite degree and the order of other vertices is independently distributed by

$$\phi(k, K) = \frac{1}{w_c(K)}(k-1)w_k, \qquad k \geq 2. \tag{2.101}$$

The distribution of the distance between the root and the vertex of infinite degree is given by

$$p(\ell, K) = \left(1 - \frac{w_c(K)}{w_1}\right)\left(\frac{w_c(K)}{w_1}\right)^{\ell-1}. \tag{2.102}$$

The outgrowths from the spine are independent subcritical Galton–Watson trees with offspring probabilities

$$p_n(K) = \frac{w_{n+1}}{\sum_{n=1}^{K} w_n}, \qquad 0 \leq n \leq K - 1. \tag{2.103}$$

As $N \longrightarrow \infty$ one finds that the size of the large vertex is approximately $(1 - m(K))N$ with high probability, where $m(K) < 1$ is the mean offspring probability of the Galton–Watson process. What makes the calculations easy in the condensed phase in the above models is the fact that the large vertex which emerges as $N \longrightarrow \infty$ has to

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[2] Galton–Watson processes are defined in Section 3.2.

stay on the spine due to the restriction on the order of the vertices in the outgrowths. When the cutoff on the vertex orders is removed ($K = \infty$) one obtains the tree model which is studied in the next chapter (with the difference that there are two marked vertices in the present model). In this case it is more difficult to locate the large vertex and one has to use other methods in the calculations. It is tempting to simply let $K \longrightarrow \infty$ in the above formulas to characterize the $K = \infty$ model, on the other hand it is not clear that interchanging the $K \longrightarrow \infty$ and $N \longrightarrow \infty$ limits is allowed. However the analysis in the next chapter shows that the above characterization of the condensed phase holds and one arrives at the same formulas as one would get by simply taking $K \longrightarrow \infty$.

3

Planar trees with a local action

In this chapter we study an equilibrium statistical mechanical model of planar trees with a local action. We start by defining the set of planar trees and endow it with a metric. We then introduce the model and show that it exhibits two phases, an elongated and a condensed phase. The main results are the proof of the asymptotic behaviour of the finite volume partition function in the condensed phase. This result is used to prove convergence of the finite volume measures to a measure on infinite trees. We conclude by calculating the annealed spectral dimension, with respect to the infinite volume measure, in the condensed phase.

3.1 Planar trees

In this section we define rooted planar trees and construct a metric on the set of all rooted planar trees. The definition resembles the one given in [35], however here we also allow vertices of infinite degree. We include vertices of infinite degree since they appear in the condensed phase of the random tree model in Section 3.3. In words, the planarity condition means that edges incident on a vertex are ordered. When the degree of a vertex is infinite one has many different possibilities of ordering the links and therefore the planarity condition must by carefully defined. The definitions below take care of this point, the vertices are allowed to have at most countably infinite degree and the edges are given the simplest possible ordering.

We start by introducing a sequence of pairwise disjoint, countable sets $(D_R)_{R\geq 0}$ with the properties that if $D_R = \emptyset$ then $D_S = \emptyset$ for all $S \geq R$. The sets D_0 and D_1 are defined to have only a single element. The set D_R will eventually denote the set of vertices at a distance R from the root. To introduce the edges and the planarity condition, we define orderings on each of the sets D_R and order preserving maps

$$\phi_R : D_R \longrightarrow D_{R-1}, \qquad R \geq 1 \tag{3.1}$$

which satisfy the following: For each vertex $v \in D_{R-1}$ such that $|\phi_R^{-1}(v)| = \infty$, there exists an order isomorphism

$$\psi_v : \mathbb{N} \longrightarrow \phi_R^{-1}(v)$$

where $\mathbb{N}$ has the standard ordering. If $|\phi_R^{-1}(v)| < \infty$ we define the order isomorphism $\psi_v : \{1, 2, \ldots, |\phi_R^{-1}(v)|\} \longrightarrow \phi_R^{-1}(v)$. One can show by induction on R that such orderings on D_R can be defined and that they are well–orderings. It is clear that given the ordered sets $D_R, R \geq 0$ and the order preserving maps $\phi_R, R \geq 1$ with the above properties, the maps ψ_v are unique.

Let $\tilde{\Gamma}$ be the set of all pairs of sequences $\{(D_0, D_1, D_2, \ldots), (\phi_1, \phi_2, \ldots)\}$ which satisfy the above conditions. Define an equivalence relation $\sim$ on $\tilde{\Gamma}$ by identifying the elements $\{(D_0, D_1, \ldots), (\phi_1, \phi_2, \ldots)\}$ and $\{(D'_0, D'_1, \ldots), (\phi'_1, \phi'_2, \ldots)\}$ if and only if for all $R \geq 1$ there exist order isomorphisms $\chi_R : D_R \longrightarrow D'_R$ such that $\phi'_R = \chi_{R-1} \circ \phi_R \circ \chi_R^{-1}$. Define $\Gamma := \tilde{\Gamma}/\sim$. If $\tau \in \tilde{\Gamma}$ we denote the equivalence class of τ by $[\tau]$ and call it a rooted planar tree, cf. Section 1.3.1. As a graph, the tree has a vertex set

$$V = \bigcup_{R=0}^{\infty} D_R$$

and an edge set

$$E = \{(v, \phi_R(v)) \mid v \in D_R, R \geq 1\}$$

which are independent of the representative $\{(D_0, D_1, \ldots), (\phi_1, \phi_2, \ldots)\}$ up to graph isomorphisms. The single element in D_0 is taken to be the root. In the following, all properties of trees $[\tau] \in \Gamma$ we are interested in are independent of representatives and we write τ instead of $[\tau]$. Rather than always specifying the sequences $(D_0, D_1, D_2, \ldots)$ and $(\phi_1, \phi_2, \ldots)$, we will refer to the elements in Γ with a single Greek letter, usually τ. We then write $D_R(\tau)$, $\phi_R(\cdot, \tau)$ etc. when we need more detailed information on τ.

Note that since the sets D_R, D'_R are well–ordered for all $R \geq 1$ the order isomor-

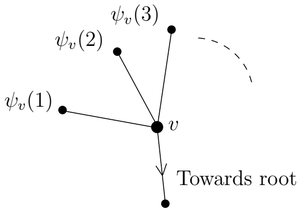


Figure 3.1: The ordering of $\phi_R^{-1}(v)$.

phisms χ_R in the above definition are unique. When we draw the trees in the plane we use the convention that $\psi_v(k)$ is the k-th vertex clockwise from the nearest neighbour of v closest to the root. See Figure 3.1.

We define the *left ball* of graph radius R, $L_R(\tau)$ as the subtree of $B_R(\tau)$ generated by subsets $E_S \subseteq D_S(B_R(\tau))$, $S = 1, ..., R$ such that $E_0 = D_0(B_R(\tau))$, $E_1 = D_1(B_R(\tau))$ and

$$E_S = \{\psi_v(i) \mid v \in E_{S-1}, i = 1, 2, \ldots, \min\{R, \sigma(v)\} - 1\} \tag{3.2}$$

for $S \geq 2$, see Fig. 3.2. It is easy to check that for all $\tau \in \Gamma$

$$|L_R(\tau)| \leq \frac{(R-1)^R - 1}{R-2}. \tag{3.3}$$

We define a metric d on Γ by

$$d(\tau_1, \tau_2) = \inf\left\{ \frac{1}{R} \;\middle|\; L_R(\tau_1) = L_R(\tau_2),\ R \in \mathbb{N} \right\}, \qquad \tau_1, \tau_2 \in \Gamma. \tag{3.4}$$

The metric used in [35,38], in the study of the generic phase, is defined in the same

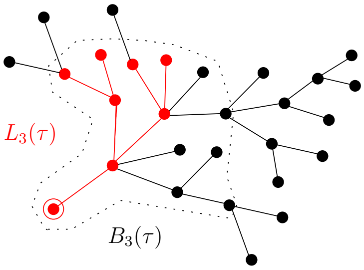


Figure 3.2: An example of the subgraphs $B_R(\tau)$ and $L_R(\tau)$.

way as d but the ball B_R is used instead of L_R. Trees which are different close to the root are always "far apart" in this metric and therefore it is only suitable to measure distances between trees of a large diameter which have vertices of finite degree. In the new metric d however, trees can be close to each other if they do not differ too much close to the root.

Denote the open ball in Γ centered at τ_0 and with radius r by

$$\mathcal{B}_r(\tau_0) = \{\tau \in \Gamma \mid d(\tau_0, \tau) < r\}. \tag{3.5}$$

In the same way as in the previous chapter on the caterpillars we find that the metric space (Γ, d) has the following properties.

Proposition 3.1.1 *The metric space* (Γ, d) *is compact.*

Proposition 3.1.2 *For* $r > 0$ *and* $\tau_0 \in \Gamma$*, the ball* $\mathcal{B}_r(\tau_0)$ *is both open and closed. Moreover, if* $\tau_1 \in \mathcal{B}_r(\tau_0)$ *then* $\mathcal{B}_r(\tau_1) = \mathcal{B}_r(\tau_0)$.

Proposition 3.1.3 *The set* Γ' *of finite trees is a countable dense subset of* Γ.

3.2 The model

Let w_n, $n \geq 1$ be a sequence of nonnegative numbers which we call *branching weights.* For technical convenience we will always take

$$w_1, w_2 > 0 \qquad \text{and} \qquad w_n > 0 \qquad \text{for some } n \geq 3. \tag{3.6}$$

Define the finite volume partition function

$$Z_N = \sum_{\tau \in \Gamma_N} \prod_{i \in V(\tau) \setminus \{r\}} w_{\sigma(i)}. \tag{3.7}$$

Define a probability distribution ν_N on Γ_N by

$$\nu_N(\tau) = Z_N^{-1} \prod_{i \in V(\tau) \setminus \{r\}} w_{\sigma(i)}. \tag{3.8}$$

The weights w_n, or alternatively the measures ν_N, define a *tree ensemble.* Note that ν_N is not affected by a rescaling of the branching weights of the form $w_n \rightarrow w_n a b^n$

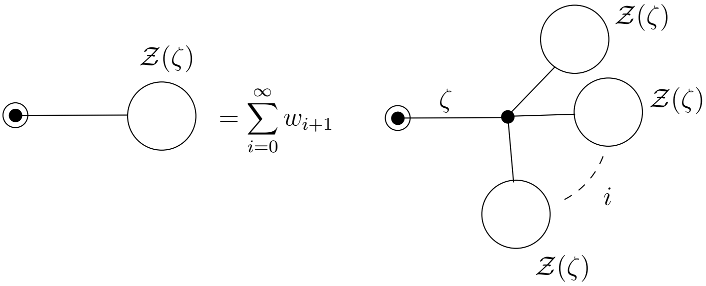


Figure 3.3: A diagram explaining the recursion (3.11). The root is indicated by a circled point.

where $a, b > 0$. We introduce the generating functions

$$\mathcal{Z}(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N \tag{3.9}$$

and

$$g(z) = \sum_{n=0}^{\infty} w_{n+1} z^n. \tag{3.10}$$

Then we have the standard relation

$$\mathcal{Z}(\zeta) = \zeta g(\mathcal{Z}(\zeta)) \tag{3.11}$$

which is explained in Fig. 3.3.

Denote the radius of convergence of $\mathcal{Z}(\zeta)$ and $g(z)$ by ζ_0 and ρ respectively and define $\mathcal{Z}_0 = \mathcal{Z}(\zeta_0)$. If $\mathcal{Z}_0 < \rho$ then we say that we have a *generic (elongated, fluid) ensemble* of trees. Otherwise we say that we have a *nongeneric ensemble.* If $\rho = \infty$ then we always have a generic ensemble. If ρ is finite then we fix $\rho = 1$ by scaling the branching weights $w_n \rightarrow w_n \rho^{n-1}$.

There is an interesting and useful relation between the tree ensemble (Γ_N, ν_N) and trees generated by the so called *Galton–Watson process.* The process is defined in the following way. We start with a single ancestor (in general there can be many) which has n offsprings with probability p_n where p_n are nonnegative numbers and

$$\sum_{n=0}^{\infty} p_n = 1. \tag{3.12}$$

Each offspring then has n offsprings itself independently with the same probabilities p_n and so on. For convenience we add a root r to the Galton–Watson trees by linking a vertex of order one to the ancestor. The process generates a probability measure on the set of all finite trees

$$\mu(\tau) = \prod_{i \in V(\tau)\setminus\{r\}} p_{\sigma(i)-1}, \qquad \text{where} \qquad \tau \in \Gamma'. \tag{3.13}$$

We define a generating function for the offspring probabilities

$$f(z) = \sum_{n=0}^{\infty} p_n z^n. \tag{3.14}$$

Galton–Watson processes are usually divided into three categories depending on the size of the first moment of the generating function $m = f'(1)$. It is clear that m represents the mean number of offsprings of each individual. If $m > 1$ the process is said to be *supercritical* and the probability that it survives forever is positive. If $m = 1$ the process is said to be *critical* and it dies out eventually with probability one. If $m < 1$ the process is said to be *subcritical* and it dies out eventually with probability one, much faster than in the critical case.

The probability distribution ν_N can be obtained from a Galton–Watson process with offspring probabilities

$$p_n = \zeta_0 w_{n+1} \mathcal{Z}_0^{n-1} \tag{3.15}$$

by conditioning the trees to be of size N

$$\nu_N(\tau) = \frac{\mu(\tau)}{\mu(\Gamma_N)}. \tag{3.16}$$

The mean offspring probability is then

$$m = \mathcal{Z}_0 \frac{g'(\mathcal{Z}_0)}{g(\mathcal{Z}_0)} \tag{3.17}$$

which we will show to be ≤ 1 by (3.11). Generic trees are always critical and nongeneric trees can be either critical or subcritical. We will now analyse this in more detail. As mentioned above $\rho = \infty$ is always generic. Let us start with a set of branching weights w_n which give $\rho = 1$. At this stage the model can be either generic or nongeneric. We fix the values of w_n for $n \geq 2$ but for now we let w_1 be a free parameter of the model. Define

$$h(\mathcal{Z}) \equiv \frac{g(\mathcal{Z})}{\mathcal{Z}}. \tag{3.18}$$

From (3.11) we see that $h(\mathcal{Z}) = 1/\zeta(\mathcal{Z})$ for $\mathcal{Z} \leq \mathcal{Z}_0$. Differentiating h we get

$$h'(\mathcal{Z}) = \frac{g(\mathcal{Z})}{\mathcal{Z}^2}\left[\mathcal{Z}\frac{g'(\mathcal{Z})}{g(\mathcal{Z})} - 1\right] \tag{3.19}$$

and again

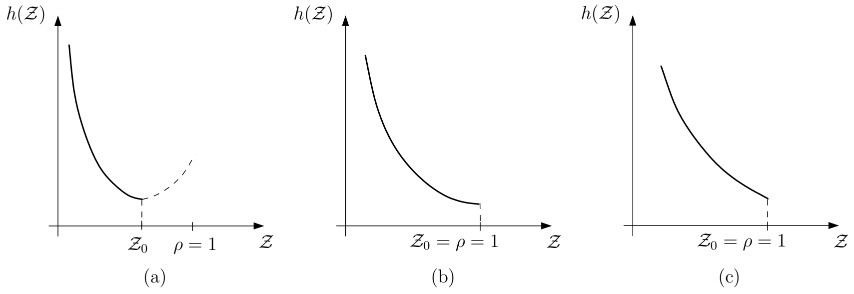
$$h''(\mathcal{Z}) = \frac{g''(\mathcal{Z})}{\mathcal{Z}} - \frac{2}{\mathcal{Z}}h'(\mathcal{Z}). \tag{3.20}$$

The genericity condition can be interpreted as h having a quadratic minimum at $\mathcal{Z} = \mathcal{Z}_0 < 1$, see Fig. 3.4. This means that $m = \mathcal{Z}_0 \frac{g'(\mathcal{Z}_0)}{g(\mathcal{Z}_0)} = 1$ showing that the generic phase corresponds to critical Galton–Watson trees. This shows that given a $\mathcal{Z}_0 < 1$ and the branching weights w_n, $n \geq 2$, it must hold that $w_1 = \sum_{n=2}^{\infty}(n-2)w_n\mathcal{Z}_0^{n-1}$. We can therefore make any model with $\rho = 1$ generic by choosing

$$w_1 < \sum_{n=2}^{\infty}(n-2)w_n \equiv w_c \tag{3.21}$$

where w_c is a critical value for w_1 which depends on w_n for $n \geq 3$. It is interesting to note that the critical value is independent of w_2. Also note that if $w_c = \infty$, i.e. if $g'(z)$ diverges as $z \longrightarrow 1$, we always have a generic ensemble.

The next possible scenario is that h has a quadratic minimum at $\mathcal{Z} = \mathcal{Z}_0 = 1$. This happens when $w_1 = w_c$ or in other words when $m = \frac{g'(1)}{g(1)} = 1$. This is a nongeneric



(a) (b) (c)

Figure 3.4: The three possible scenarios. (a) Generic, critical, $w_1 < w_c$. (b) Nongeneric, critical, $w_1 = w_c$. (c) Nongeneric, subcritical, $w_1 > w_c$.

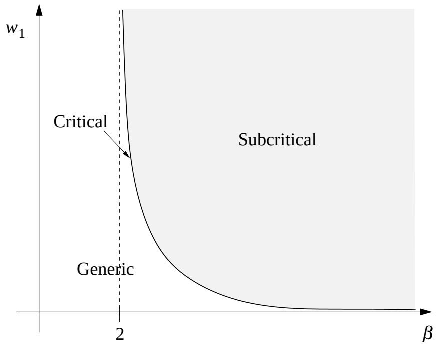


Figure 3.5: A diagram showing the possible phases of the trees. The critical line is determined by the equation $w_1 = w_c$.

ensemble which still corresponds to critical Galton–Watson trees.

Finally, by choosing $w_1 > w_c$, h has no quadratic minimum and $m = \frac{g'(1)}{g(1)} < 1$. In this case the trees are nongeneric and correspond to subcritical Galton–Watson trees. We will refer to this phase as the *subcritical nongeneric phase* or the *condensed phase*.

3.3 Subcritical nongeneric trees

In this section we examine the subcritical nongeneric phase and determine the asymptotic behaviour of Z_N. We fix a number β and for $n \geq 2$ we fix the branching weights such that

$$w_n = n^{-\beta}(1 + o(1)), \quad n \geq 2 \tag{3.22}$$

and for now w_1 is a free parameter. In this case $\rho = 1$. If $\beta \leq 2$ then $g'(1) = \infty$ and therefore we are in the generic phase for all values of w_1. If $\beta > 2$ we can have any one of the three cases discussed in the previous section depending on the value of w_1, see Fig. 3.5. Now choose $\beta > 2$ and $w_1 > w_c$ such that

$$m = \frac{g'(1)}{g(1)} < 1, \tag{3.23}$$

meaning we are in the nongeneric, subcritical phase. Then $\mathcal{Z}_0 = \rho = 1$ and we see from (3.11) that

$$\zeta_0 = \frac{1}{g(1)}. \tag{3.24}$$

The main result of this section is the following.

Theorem 3.3.1 *For the branching weights (3.22) which satisfy (3.23) it holds that*

$$Z_N = (1-m)^{-\beta} N^{-\beta} \zeta_0^{1-N} \left(1 + o(1)\right). \qquad (3.25)$$

To determine the large N behaviour of Z_N we split it into the following sum

$$Z_N = Z_{1,N} + E_N \qquad (3.26)$$

where $Z_{1,N}$ is the contribution to Z_N from trees which have exactly 1 vertex of maximum degree and E_N is the contribution to Z_N from trees which have ≥ 2 vertices of maximum degree. The plan is to estimate these two terms separately and show that for large N the main contribution is from $Z_{1,N}$. It will follow from the proof that large trees, of size N, are most likely to have exactly one large vertex which is approximately of degree $(1-m)N$. This will be stated more precisely in Section 3.4. The arguments used in the proof of Theorem 3.3.1 rely on a "truncation method" and some classical results from probability theory. We begin the proof by defining truncated versions of the generating functions introduced in the previous section. Then we introduce notation from probability theory and state a few lemmas. In Subsection 3.3.1 we analyse the asymptotic behaviour of $Z_{1,N}$ and in Subsection 3.3.2 we do the same for E_N.

For the truncation method, we will need the following definitions. Let $L_{i,N}$ be the finite volume partition function for trees of N edges which have all vertices of degree $\leq i$ and define the functions

$$\mathcal{L}_i(\zeta) = \sum_{N=1}^{\infty} L_{i,N} \zeta^N \qquad (3.27)$$

and

$$\ell_i(z) = \sum_{n=0}^{i-1} w_{n+1} z^n. \qquad (3.28)$$

We have the standard relation

$$\mathcal{L}_i(\zeta) = \zeta \ell_i(\mathcal{L}_i(\zeta)) \qquad (3.29)$$

obtained in the same way as (3.11).

Let $Y_{j,i,N}$ be the finite volume partition function for trees of N edges which have all vertices of degree $\leq i$ and one marked (not weighted) vertex of degree one at distance j from the root. Define

$$\mathcal{Y}_{j,i}(\zeta) = \sum_{N=1}^{\infty} Y_{j,i,N}\zeta^N \tag{3.30}$$

and

$$\mathcal{Y}_i(\zeta) = \sum_{j=1}^{\infty} \mathcal{Y}_{j,i}(\zeta). \tag{3.31}$$

With generating function arguments we find that

$$\mathcal{Y}_{j,i}(\zeta) = \zeta \ell_i'(\mathcal{L}_i(\zeta))\mathcal{Y}_{j-1,i}(\zeta) \tag{3.32}$$

for $j \geq 2$, see Fig. 3.6. Using $\mathcal{Y}_{1,i}(\zeta) = \zeta$ this yields

$$\mathcal{Y}_{j,i}(\zeta) = \zeta\Big(\zeta\ell_i'(\mathcal{L}_i(\zeta))\Big)^{j-1}, \tag{3.33}$$

and by summing over j we get

$$\mathcal{Y}_i(\zeta) = \frac{\zeta}{1-\zeta\ell_i'(\mathcal{L}_i(\zeta))}. \tag{3.34}$$

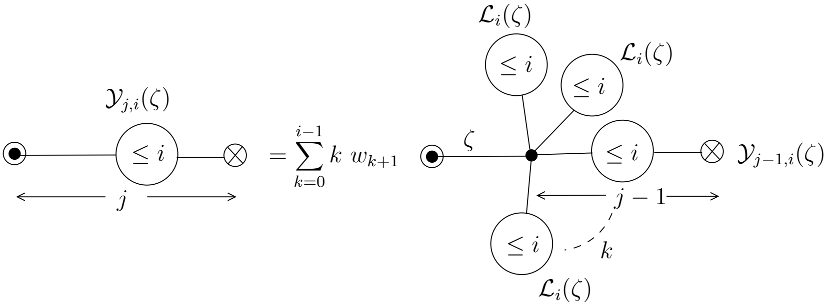


Figure 3.6: A diagram explaining (3.33). The marked vertex is indicated by ⊗. The balloons which include the "$\leq i$" are trees which have vertices of degree at most i. If the degree of the nearest neighbour to the root is $k+1$, there are k different ways of placing the marked vertex onto a balloon.

It will be useful to formulate our problem in the language of probability theory. Define the probability generating functions

$$f_i(z) = \frac{\ell_i(z)}{\ell_i(1)} \qquad \text{and} \qquad f(z) = \frac{g(z)}{g(1)}. \tag{3.35}$$

Let $X_1^{(i)}, X_2^{(i)}, \ldots$ be i.i.d. random variables which have a probability generating function $f_i(z)$ i.e.

$$\mathbb{P}(X_j^{(i)} = k) = \begin{cases} w_{k+1}/\ell_i(1) & \text{if } 0 \leq k \leq i-1, \\ 0 & \text{if } k > i-1, \end{cases} \tag{3.36}$$

and let $X_1, X_2, \ldots$ be i.i.d. random variables which have a probability generating function $f(z)$. Define

$$m_i = \mathbb{E}(X_j^{(i)}), \quad \sigma_i^2 = \mathrm{Var}(X_j^{(i)}), \qquad S_N^{(i)} = X_1^{(i)} + \ldots + X_N^{(i)} \tag{3.37}$$

and

$$S_N = X_1 + \ldots + X_N. \tag{3.38}$$

Note that $m = \mathbb{E}(X_j)$ and from (3.23) we know that $m < 1$. Clearly $m_i \longrightarrow m$ as $i \longrightarrow \infty$. We need the following lemmas, the first three deal with convergence rates in the weak law of large numbers.

Lemma 3.3.2 *For any $\epsilon > 0$ and any $s < \beta - 2$ it holds that*

$$\lim_{N\to\infty} N^s \mathbb{P}\left(\left|\frac{S_N}{N} - m\right| > \epsilon\right) = 0. \tag{3.39}$$

Proof It is clear that $\mathbb{E}(|X_j|^t) < \infty$ for all $t < \beta - 1$. The result then follows directly from [60, Theorem 28, pg. 286].

□

The next Lemma is a classical result of Bennett [15].

Lemma 3.3.3 *(Bennett's inequality) If $W_1, W_2, \ldots$ are independent random variables, $\mathbb{E}(W_j) = 0$, $\mathrm{Var}(W_j) = \sigma_W^2$ and $W_j \leq b$ a.s. for every j, where b and σ_W are positive numbers, then for any $\epsilon > 0$*

$$\mathbb{P}\left(\frac{1}{N}\sum_{j=1}^{N} W_j > \epsilon\right) \leq \exp\left\{-\eta\left[\left(1 + \frac{1}{\lambda}\right)\log\left(1+\lambda\right) - 1\right]\right\} \tag{3.40}$$

with

$$\eta = \frac{N\epsilon}{b} \qquad \text{and} \qquad \lambda = \frac{b\epsilon}{\sigma_W^2}. \tag{3.41}$$

Lemma 3.3.4 *If $i = O(N^\gamma)$ where $\gamma < 1$ then for any $\epsilon > 0$ small enough the following holds*

$$\mathbb{P}\left(\frac{S_N^{(i)}}{N} - m_i > \epsilon\right) \leq \exp\left\{-C\epsilon N^{1-\gamma}\right\}. \tag{3.42}$$

where C is a positive constant.

Proof This follows directly from Bennett's inequality with $W_j = X_j^{(i)} - m_i$. Then $\sigma_W = \sigma_i$ and we can take $b = i$ since $X_j^{(i)} < i$ almost surely. Now assume that $i = O(N^\gamma)$. Then

$$\eta = \epsilon O(N^{1-\gamma}). \tag{3.43}$$

If $\beta > 3$ then $\sigma_i < \infty$ and $\lambda = O(N^\gamma)$ and the result follows. If $2 < \beta \leq 3$ then

$$\sigma_i^2 = \begin{cases} O(i^{3-\beta}) & \text{if } \beta < 3, \\ O(\log(i)) & \text{if } \beta = 3 \end{cases} \tag{3.44}$$

so $\lambda \longrightarrow \infty$ as $N \longrightarrow \infty$ which completes the proof.

□

In the following we will use Lagrange's inversion formula repeatedly [66, pg. 167]. We understand $[z^n]\{f(z)\}$ as the coefficient of z^n in the Taylor expansion of f about $z = 0$.

Lemma 3.3.5 *(Lagrange's inversion formula) If $h(z)$ is a formal power series in z and $\mathcal{L}_i$ satisfies (3.29) then*

$$[\zeta^N]\{h(\mathcal{L}_i(\zeta))\} = \frac{1}{N}[z^{N-1}]\left\{h'(z)\ell_i(z)^N\right\}. \tag{3.45}$$

Applying the above to the function $h(z) = z^j$ we get

$$[\zeta^N]\left\{\mathcal{L}_i(\zeta)^j\right\} = \frac{j}{N}[z^{N-j}]\left\{\ell_i(z)^N\right\}. \tag{3.46}$$

The following lemma will be helpful. We omit the proof since it is trivial.

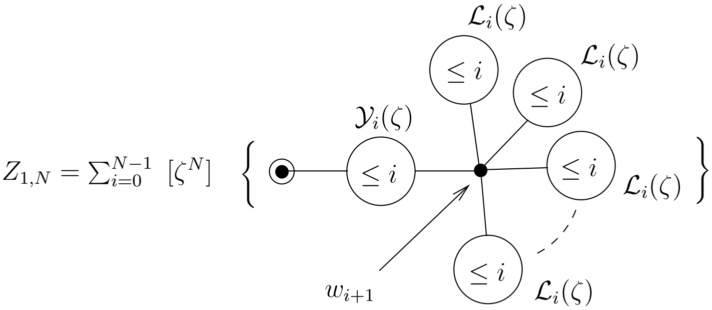


Figure 3.7: An illustration of Equation (3.47). The balloons which include the "$\leq i$" are trees which have vertices of degree at most i. There is thus precisely one vertex of maximum degree $i+1$.

Lemma 3.3.6 *If $X \geq 0$ and Y are random variables then for any $\epsilon > 0$*

$$\mathbb{P}\left(|X+Y| \leq \epsilon\right) \geq \mathbb{P}\left(X \leq \epsilon/2\right) \mathbb{P}\left(|Y| \leq \epsilon/2\right)$$

and

$$\mathbb{P}\left(|X+Y| > \epsilon\right) \leq \mathbb{P}\left(|Y| > \epsilon/2\right) + \mathbb{P}\left(X > \epsilon/2\right).$$

3.3.1 Calculation of $Z_{1,N}$

Using the lemmas in the previous subsection we are ready to study the asymptotic behaviour of $Z_{1,N}$. It is easy to see that

$$Z_{1,N} = \sum_{i=0}^{N-1} w_{i+1} [\zeta^N] \left\{\mathcal{Y}_i(\zeta) \mathcal{L}_i(\zeta)^i\right\}, \tag{3.47}$$

as is explained in Fig. 3.7. Combining Equations (3.29) and (3.34) one can use the Lagrange inversion formula (3.45) for the function

$$h_{ij}(z) = \frac{z^{j+1}}{\ell_i(z) - z\ell_i'(z)} \tag{3.48}$$

to get

$$[\zeta^N] \left\{\mathcal{Y}_i(\zeta) \mathcal{L}_i(\zeta)^j\right\} = \frac{1}{N} [z^{N-j-1}] \left\{ \left(\frac{j+1}{\ell_i(z) - z\ell_i'(z)} + \frac{z^2 \ell_i''(z)}{(\ell_i(z) - z\ell_i'(z))^2} \right) \ell_i(z)^N \right\}.$$

Note that the left hand side is increasing in i and therefore of course also the right hand side. This fact will be used repeatedly in the proof of Lemma 3.3.8. Define the functions

$$f_{i,1}(z) = \frac{\ell_i(1) - \ell_i'(1)}{\ell_i(z) - z\ell_i'(z)} \tag{3.49}$$

and

$$f_{i,2}(z) = \frac{z^2\ell_i''(z)}{(\ell_i(z) - z\ell_i'(z))^2}\frac{(\ell_i(1) - \ell_i'(1))^2}{\ell_i''(1)}. \tag{3.50}$$

It is easy to check that all derivatives of the functions are positive for $0 \leq z \leq 1$ and that $f_{i,1}(1) = f_{i,2}(1) = 1$. We then define $X^{(i,1)}$ and $X^{(i,2)}$ to be random variables having $f_{i,1}$ and $f_{i,2}$, respectively, as probability generating functions. We will need the following lemma

Lemma 3.3.7 *If* $i = O(N)$ *as* $N \longrightarrow \infty$ *then for any* $\epsilon > 0$

1. $\mathbb{P}\left(X^{(i,1)} \geq \epsilon N\right) \leq C_1 N^{2-\beta}$,

2. $\ell_N''(1)\mathbb{P}\left(X^{(i,2)} \geq \epsilon N\right) \leq C_2 \begin{cases} N^{3-\beta} & \text{if } \beta \neq 3, \\ \log(N) & \text{if } \beta = 3, \end{cases}$

where C_1 *and* C_2 *are positive numbers which in general depend on* ϵ *and* β.

Proof We use a weighted version of Chebyshev's inequality which states that if X is a random variable and $\phi(x) > 0$ for $x > 0$ is monotonically increasing and $\mathbb{E}(\phi(X))$ exists then

$$\mathbb{P}\left(|X| \geq t\right) \leq \frac{\mathbb{E}(\phi(X))}{\phi(t)}. \tag{3.51}$$

First we prove case (1). Choose $\phi(x) = x^{\lfloor \beta \rfloor}$ where $\lfloor \cdot \rfloor$ denotes the floor function. It is clear that $f_{i,1}^{(n)}(1) < \infty$ for all n and therefore $\mathbb{E}(\phi(X^{(i,1)})) < \infty$. One can check that as $i \longrightarrow \infty$

$$\mathbb{E}(\phi(X^{(i,1)})) = O(\ell_i^{(\lfloor \beta \rfloor + 1)}(1)) = O(i^{-\beta + \lfloor \beta \rfloor + 2}). \tag{3.52}$$

If $i = O(N)$ as $N \longrightarrow \infty$ then by the Chebyshev inequality there exists a positive constant C such that

$$\mathbb{P}\left(X^{(i,1)} \geq \epsilon N\right) \leq C\frac{N^{-\beta + \lfloor \beta \rfloor + 2}}{(\epsilon N)^{\lfloor \beta \rfloor}} = C\frac{N^{2-\beta}}{\epsilon^{\lfloor \beta \rfloor}}. \tag{3.53}$$

In the proof of case (2) first consider the case when $2 < \beta \leq 3$. Then

$$\ell_N''(1) = \begin{cases} O(N^{3-\beta}) & \text{if } \beta \neq 3, \\ O(\log(N)) & \text{if } \beta = 3, \end{cases} \tag{3.54}$$

as $N \longrightarrow \infty$ which proves the claim. If $\beta > 3$ then $\ell_N''(1)$ is finite when $N \longrightarrow \infty$ and the proof is exactly the same as in case (1).

□

We are now ready to prove the main result of this subsection.

Lemma 3.3.8

$$Z_{1,N} = (1-m)^{-\beta} N^{-\beta} \zeta_0^{1-N} \left(1 + o(1)\right). \tag{3.55}$$

Proof In this proof we let $C, C_1, C_2, \ldots$ denote positive numbers independent of N whose values may differ between equations. Define

$$\begin{aligned} G_N(a,b) &= g(1)^{1-N} N^{\beta-1} \sum_{a \leq n \leq b} w_{N-n} [z^n] \Bigg\{ \ell_{N-n-1}(z)^N \\ &\times \left(\frac{N-n}{\ell_{N-n-1}(z) - z\ell_{N-n-1}'(z)} + \frac{z^2 \ell_{N-n-1}''(z)}{(\ell_{N-n-1}(z) - z\ell_{N-n-1}'(z))^2} \right) \Bigg\}. \end{aligned} \tag{3.56}$$

It follows that

$$N^\beta \zeta_0^{N-1} Z_{1,N} = G_N(0, N-1). \tag{3.57}$$

Now choose an $\epsilon > 0$ small enough and a γ such that $2/\beta < \gamma < 1$ and split the above expression into four terms

$$\begin{aligned} N^\beta \zeta_0^{N-1} Z_{1,N} &= G_N(0, \lfloor (m-\epsilon)N \rfloor) + G_N(\lfloor (m-\epsilon)N \rfloor + 1, \lfloor (m+\epsilon)N \rfloor) \\ &+ G_N(\lfloor (m+\epsilon)N \rfloor + 1, \lfloor N - N^\gamma \rfloor) + G_N(\lfloor N - N^\gamma \rfloor + 1, N-1). \end{aligned} \tag{3.58}$$

We show that as $N \longrightarrow \infty$ and $\epsilon \longrightarrow 0$ the second term has a positive limit but the other terms converge to zero. To make the notation more compact define

$$N_+ = N - \lfloor (m+\epsilon)N \rfloor - 1 \qquad \text{and} \qquad N_- = N - \lfloor (m-\epsilon)N \rfloor. \tag{3.59}$$

The first term in (3.58) can be estimated from above by

$$\begin{aligned}
&G_N(0, \lfloor (m-\epsilon)N \rfloor) \\
&\quad \leq \left(\frac{N}{N_-}\right)^{\beta-1} \sum_{n=0}^{\lfloor (m-\epsilon)N \rfloor} [z^n] \left\{ f(z)^N \left(C_1 f_{N,1}(z) + \frac{C_2 \ell_N''(1)}{N-n} f_{N,2}(z) \right) \right\} \\
&\quad \leq C_3 \mathbb{P}\left(\left| \frac{S_N + X^{(N,1)}}{N} - m \right| > \epsilon \right) + \frac{C_4 \ell_N''(1)}{N} \mathbb{P}\left(\left| \frac{S_N + X^{(N,2)}}{N} - m \right| > \epsilon \right).
\end{aligned} \tag{3.60}$$

By Lemma 3.3.6 we have for $i = 1, 2$,

$$\mathbb{P}\left(\left| \frac{S_N + X^{(N,i)}}{N} - m \right| > \epsilon \right) \leq \mathbb{P}\left(\left| \frac{S_N}{N} - m \right| > \epsilon/2 \right) + \mathbb{P}\left(X^{(N,i)} > N\epsilon/2 \right). \tag{3.61}$$

This, combined with Equation (3.54) and Lemmas 3.3.2 and 3.3.7, shows that the two terms in (3.60) go to zero as $N \longrightarrow \infty$.

The third term in (3.58) is estimated from above by

$$\begin{aligned}
&G_N(\lfloor (m+\epsilon)N \rfloor + 1, \lfloor N - N^\gamma \rfloor) \leq \left(\frac{N}{N - \lfloor N - N^\gamma \rfloor} \right)^{\beta-1} \\
&\quad \times \sum_{n=\lfloor (m+\epsilon)N \rfloor + 1}^{\lfloor N - N^\gamma \rfloor} [z^n] \left\{ f(z)^N \left(C_1 f_{N,1}(z) + \frac{C_2 \ell_N''(1)}{(N - \lfloor N - N^\gamma \rfloor)} f_{N,2}(z) \right) \right\} \\
&\leq C_3 N^{(1-\gamma)(\beta-1)} \mathbb{P}\left(\left| \frac{S_N + X^{(N,1)}}{N} - m \right| > \epsilon \right) \\
&\quad + C_4 N^{(1-\gamma)(\beta-1)-\gamma} \ell_N''(1) \mathbb{P}\left(\left| \frac{S_N + X^{(N,2)}}{N} - m \right| > \epsilon \right).
\end{aligned} \tag{3.62}$$

Since $\gamma > 2/\beta$ it holds that $(1-\gamma)(\beta-1) < \beta - 2$ and $(1-\gamma)(\beta-1) - \gamma < \beta - 3$. Then by (3.54), (3.61) and Lemmas 3.3.2 and 3.3.7 we see that last two terms converge to zero as $N \longrightarrow \infty$.

To estimate the fourth term of (3.58) from the above we first note that

$$[\zeta^N] \{\mathcal{Y}_i(\zeta)\} = [\zeta^N] \left\{ \frac{\partial}{\partial w_1} \mathcal{L}_i(\zeta) \right\} \leq \frac{N}{w_1} [\zeta^N] \{\mathcal{L}_i(\zeta)\} \tag{3.63}$$

and thus

$$G_N(a,b) \leq w_1^{-1} g(1)^{1-N} N^{\beta} \sum_{a \leq n \leq b} w_{N-n}(N-n)[z^n]\left\{\ell_{N-n-1}(z)^N\right\}. \tag{3.64}$$

Using this for N large enough and ϵ small enough we get

$$\begin{aligned}
G_N(\lfloor N-N^{\gamma}\rfloor+1, N-1) &\leq C_1 N^{\beta} \sum_{n=\lfloor N-N^{\gamma}\rfloor+1}^{N-1} [z^n]\left\{f_{N-\lfloor N-N^{\gamma}\rfloor}(z)^N\right\} \\
&\leq C_1 N^{\beta} \mathbb{P}\left(\frac{S_N^{(N-\lfloor N-N^{\gamma}\rfloor)}}{N} - m_{N-\lfloor N-N^{\gamma}\rfloor} \geq \epsilon\right) \\
&\leq C_1 N^{\beta} \exp\left(-C_2 \epsilon N^{1-\gamma}\right) \qquad (3.65)
\end{aligned}$$

where in the last step we used Lemma 3.3.4. The last expression converges to zero as $N \longrightarrow \infty$ since $\gamma < 1$.

Finally we show that the second term in (3.58) has a nonzero contribution as $N \longrightarrow \infty$. By (3.22) we can choose n large enough such that

$$(1-\epsilon)n^{-\beta} \leq w_n \leq (1+\epsilon)n^{-\beta}.$$

We then get the upper bound

$$\begin{aligned}
&G_N(\lfloor (m-\epsilon)N\rfloor+1, \lfloor (m+\epsilon)N\rfloor) \leq (1+\epsilon)g(1)\left(\frac{N}{N_+}\right)^{\beta-1} \\
&\times \left(\frac{1}{\ell_N(1)-\ell_N'(1)} \sum_{n=\lfloor (m-\epsilon)N\rfloor+1}^{\lfloor (m+\epsilon)N\rfloor} [z^n]\left\{f_{N,1}(z) f(z)^N\right\}\right. \\
&\left. + \frac{\ell_N''(1)}{(\ell_N(1)-\ell_N'(1))^2 N_+} \sum_{n=\lfloor (m-\epsilon)N\rfloor+1}^{\lfloor (m+\epsilon)N\rfloor} [z^n]\left\{f_{N,2}(z) f(z)^N\right\}\right) \\
&\leq (1+\epsilon)g(1)\left(\frac{N}{N_+}\right)^{\beta-1}\left(\frac{1}{\ell_N(1)-\ell_N'(1)} + \frac{\ell_N''(1)}{(\ell_N(1)-\ell_N'(1))^2 N_+}\right) \\
&\longrightarrow \frac{(1+\epsilon)(1-(m+\epsilon))^{1-\beta}}{1-m} \qquad (3.66)
\end{aligned}$$

as $N \longrightarrow \infty$ by (3.54). In a similar way we get the lower bound

$$
\begin{aligned}
G_N(\lfloor (m-\epsilon)N \rfloor + 1, \lfloor (m+\epsilon)N \rfloor) \geq (1-\epsilon) g(1) \left(\frac{N}{N_-}\right)^{\beta-1} \left(\frac{\ell_{N_+}(1)}{g(1)}\right)^N \\
\times \left(\frac{1}{\ell_{N_+}(1) - \ell'_{N_+}(1)} \sum_{n=\lfloor (m-\epsilon)N \rfloor + 1}^{\lfloor (m+\epsilon)N \rfloor} [z^n] \left\{ f_{N_+,1}(z) f_{N_+}(z)^N \right\} \right. \\
\left. + \frac{\ell''_{N_+}(1)}{(\ell_{N_+}(1) - \ell'_{N_+}(1))^2 N_-} \sum_{n=\lfloor (m-\epsilon)N \rfloor + 1}^{\lfloor (m+\epsilon)N \rfloor} [z^n] \left\{ f_{N_+,2}(z) f_{N_+}(z)^N \right\} \right).
\end{aligned}
\tag{3.67}
$$

By (3.54) the second term converges to zero as $N \longrightarrow \infty$. Looking at the first term we find that

$$
\frac{(1-\epsilon) g(1)}{\ell_{N_+}(1) - \ell'_{N_+}(1)} \left(\frac{N}{N_-}\right)^{\beta-1} \longrightarrow \frac{(1-\epsilon)(1-(m-\epsilon))^{1-\beta}}{1-m} \tag{3.68}
$$

as $N \longrightarrow \infty$ and

$$
\left(\frac{\ell_{N_+}(1)}{g(1)}\right)^N = \left(1 - \frac{1}{g(1)} \sum_{n=N_+}^{\infty} w_{n+1}\right)^N = \left(1 + O(N^{-\beta+1})\right)^N \longrightarrow 1 \tag{3.69}
$$

as $N \longrightarrow \infty$ since $\beta > 2$. Finally we have for N large enough

$$
\begin{aligned}
\sum_{n=\lfloor (m-\epsilon)N \rfloor + 1}^{\lfloor (m+\epsilon)N \rfloor} [z^n] \left\{ f_{N_+,1}(z) f_{N_+}(z)^N \right\} &= \mathbb{P}\left(\left| \frac{S_N^{(N_+)} + X^{(N_+,1)}}{N} - m \right| \leq \epsilon \right) \\
&\geq \mathbb{P}\left(\left| \frac{S_N^{(N_+)} + X^{(N_+,1)}}{N} - m_{N_+} \right| \leq \epsilon/2 \right) \\
&\geq \mathbb{P}\left(\left| \frac{S_N^{(N_+)}}{N} - m_{N_+} \right| \leq \epsilon/4 \right) \mathbb{P}\left(X^{(N_+,1)} \leq N\epsilon/4 \right) \\
&\geq \left(1 - \frac{\sigma_{N_+}^2}{N\,(\epsilon/4)^2} \right) \left(1 - C N^{2-\beta} \right)
\end{aligned}
\tag{3.70}
$$

where in the second last step we used Lemma 3.3.6 and in the last step we used Chebyshev's inequality and Lemma 3.3.7. It is clear from (3.44) that $\sigma_{N_+}^2/N \longrightarrow 0$ as $N \longrightarrow \infty$ and therefore the last expression converges to 1.

From the above estimates (3.60), (3.62) and (3.65–3.70) we find that

$$\begin{aligned}\frac{(1-\epsilon)\left(1-(m-\epsilon)\right)^{1-\beta}}{1-m} &\leq \liminf_{N\to\infty} N^{\beta}\zeta_0^{N-1}Z_{1,N} \\ &\leq \limsup_{N\to\infty} N^{\beta}\zeta_0^{N-1}Z_{1,N} \leq \frac{(1+\epsilon)\left(1-(m+\epsilon)\right)^{1-\beta}}{1-m}.\end{aligned}$$

Since this holds for all $\epsilon > 0$ small enough, the limit exists and

$$\lim_{N\to\infty} N^{\beta}\zeta_0^{N-1}Z_{1,N} = (1-m)^{-\beta} \tag{3.71}$$

which completes the proof.

□

3.3.2 An estimate of E_N

We now estimate E_N, the remaining contribution to Z_N. Note that $\mathcal{L}_{i+1}(\zeta) - \mathcal{L}_i(\zeta)$ is the grand canonical partition function for trees which have at least one vertex of degree $i+1$ and no vertex of degree greater than $i+1$. Consider a tree which has at least 2 vertices of max degree $i+1$. Denote the two max degree vertices closest to the root and second closest to the root by s_1 and s_2 respectively. They are not necessarily unique but for the following purpose we can choose any we like. Denote the path from the root to s_2 by (r, s_2). We can write

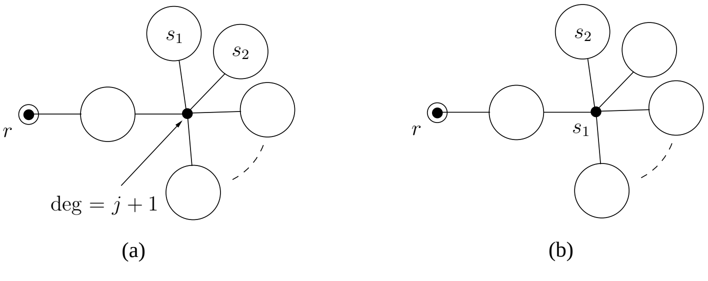


Figure 3.8: (a) The case when $s_1 \notin (r, s_2)$. At least two balloons attached to the vertex of degree $j+1$ (excluding the rooted one) indicated in the figure have to have at least one vertex of degree $i+1$, namely s_1 and s_2. (b) The case when $s_1 \in (r, s_2)$. At least one balloon attached to the vertex s_1 (excluding the rooted one) has to have at least one vertex of degree $i+1$, namely s_2.

$$\begin{aligned} E_N \;=\; & \sum_{i=0}^{\lfloor \frac{N+1}{2} \rfloor - 1} \left(\sum_{j=0}^{i-1} w_{j+1}[\zeta^N] \left\{ \mathcal{Y}_i(\zeta) \sum_{n=2}^{j} \binom{j}{n} \underbrace{\left(\mathcal{L}_{i+1}(\zeta) - \mathcal{L}_i(\zeta)\right)^n}_{s_1 \text{ and } s_2 \text{ in here}} \mathcal{L}_i(\zeta)^{j-n} \right\} \right. \\ & \left. + \underbrace{w_{i+1}}_{s_1}[\zeta^N] \left\{ \mathcal{Y}_i(\zeta) \sum_{n=1}^{i} \binom{i}{n} \underbrace{\left(\mathcal{L}_{i+1}(\zeta) - \mathcal{L}_i(\zeta)\right)^n}_{s_2 \text{ in here}} \mathcal{L}_i(\zeta)^{i-n} \right\} \right). \end{aligned} \tag{3.72}$$

The outermost sum is over all possible max degrees. The first term in the brackets takes care of the case when $s_1 \notin (r, s_2)$. Then $j+1$ is the degree of the vertex where (r, s_1) and (r, s_2) start to differ. At least two of the subtrees attached to this vertex (excluding the rooted one) have to have at least one vertex of degree $i+1$. See Figure 3.8 (a). The second term in the brackets takes care of the case when $s_1 \in (r, s_2)$. At least one of the subtrees attached to s_1 (excluding the rooted one) has to have at least one vertex of degree $i+1$. See Figure 3.8 (b).

Lemma 3.3.9

$$[\zeta^N]\left\{\mathcal{L}_{i+1}(\zeta) - \mathcal{L}_i(\zeta)\right\} \leq \frac{w_{i+1}N}{i}[\zeta^N]\left\{\zeta \mathcal{L}_{i+1}(\zeta)^i\right\}. \tag{3.73}$$

Proof Use the Lagrange inversion theorem to obtain

$$\begin{aligned} [\zeta^N]\left\{\mathcal{L}_{i+1}(\zeta) - \mathcal{L}_i(\zeta)\right\} &= \frac{1}{N}[z^{N-1}]\left\{\ell_{i+1}(z)^N - \ell_i(z)^N\right\} \\ &= \frac{1}{N}[z^{N-1}]\left\{(\ell_{i+1}(z) - \ell_i(z)) \sum_{N_1+N_2=N-1} \ell_{i+1}(z)^{N_1}\ell_i(z)^{N_2}\right\} \\ &\leq w_{i+1}[z^{N-i-1}]\left\{\ell_{i+1}(z)^{N-1}\right\} \end{aligned}$$

Now use the Lagrange inversion theorem the opposite way to obtain the result.

□

Lemma 3.3.10

$$E_N \leq 2N^2 \sum_{i=0}^{N-1} w_{i+1}^2[\zeta^{N-1}]\left\{\mathcal{Y}_{i+1}(\zeta)\mathcal{L}_{i+1}(\zeta)^{2i-1}\right\}. \tag{3.74}$$

Proof First note that

$$\begin{aligned}
&\sum_{n=2}^{j}\binom{j}{n}\left(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta)\right)^n\mathcal{L}_i(\zeta)^{j-n}\\
=\ &\mathcal{L}_{i+1}(\zeta)^j-\mathcal{L}_i(\zeta)^j-j(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta)\mathcal{L}_i(\zeta)^{j-1}\\
=\ &(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta))\left(\sum_{j_1+j_2=j-1}\mathcal{L}_{i+1}(\zeta)^{j_1}\mathcal{L}_i(\zeta)^{j_2}-j\mathcal{L}_i(\zeta)^{j-1}\right)\\
\leq\ &j(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta))\left(\mathcal{L}_{i+1}(\zeta)^{j-1}-\mathcal{L}_i(\zeta)^{j-1}\right)\\
=\ &j(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta))^2\sum_{j_1+j_2=j-2}\mathcal{L}_{i+1}(\zeta)^{j_1}\mathcal{L}_i(\zeta)^{j_2}\\
\leq\ &j(j-1)(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta))^2\mathcal{L}_{i+1}(\zeta)^{j-2}.
\end{aligned}$$

It is also clear that the above inequality holds inside $[\zeta^N]\{\cdot\}$ brackets. Therefore the sum over j in (3.72) is estimated from above by

$$\begin{aligned}
&\sum_{j=0}^{i-1}w_{j+1}[\zeta^N]\left\{\mathcal{Y}_i(\zeta)\sum_{n=2}^{j}\binom{j}{n}\left(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta)\right)^n\mathcal{L}_i(\zeta)^{j-n}\right\}\\
\leq\ &[\zeta^N]\left\{\mathcal{Y}_i(\zeta)(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta))^2\ell_i''(\mathcal{L}_{i+1}(\zeta))\right\}.
\end{aligned}$$

Now use Lemma 3.3.9 to get

$$\begin{aligned}
&[\zeta^N]\left\{\mathcal{Y}_i(\zeta)(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta))^2\ell_i''(\mathcal{L}_{i+1}(\zeta))\right\}\\
=\ &\sum_{N_1+N_2+N_3=N}[\zeta^{N_1}]\left\{\mathcal{Y}_i(\zeta)\ell_i''(\mathcal{L}_{i+1}(\zeta))\right\}[\zeta^{N_2}]\left\{\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta)\right\}[\zeta^{N_3}]\left\{\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta)\right\}\\
\leq\ &\frac{w_{i+1}^2}{i^2}N^2\sum_{N_1+N_2+N_3=N}[\zeta^{N_1}]\left\{\mathcal{Y}_i(\zeta)\ell_i''(\mathcal{L}_{i+1}(\zeta))\right\}[\zeta^{N_2}]\left\{\zeta\mathcal{L}_{i+1}(\zeta)^i\right\}[\zeta^{N_3}]\left\{\zeta\mathcal{L}_{i+1}(\zeta)^i\right\}\\
=\ &\frac{w_{i+1}^2}{i^2}N^2[\zeta^N]\left\{\zeta^2\mathcal{Y}_i(\zeta)\ell_i''(\mathcal{L}_{i+1}(\zeta))\mathcal{L}_{i+1}(\zeta)^{2i}\right\}.
\end{aligned}$$

Observe that

$$\frac{\zeta\ell_i''(\mathcal{L}_{i+1}(\zeta))\mathcal{L}_{i+1}(\zeta)}{i^2}\leq\frac{\zeta\ell_{i+1}(\mathcal{L}_{i+1}(\zeta))}{\mathcal{L}_{i+1}(\zeta)}=1 \tag{3.75}$$

where the last equality follows from (3.29). Combining the above results we get the estimate

$$\begin{aligned}
&\sum_{j=0}^{i-1} w_{j+1}[\zeta^N]\left\{\mathcal{Y}_i(\zeta)\sum_{n=2}^{j}\binom{j}{n}\left(\mathcal{L}_{i+1}(\zeta)-\mathcal{L}_i(\zeta)\right)^n \mathcal{L}_i(\zeta)^{j-n}\right\} \\
\leq\ & w_{i+1}^2 N^2[\zeta^{N-1}]\left\{\mathcal{Y}_{i+1}(\zeta)\mathcal{L}_{i+1}(\zeta)^{2i-1}\right\}.
\end{aligned}$$

We get precisely the same estimate for the term in the second line in (3.72) (the calculations are even simpler) except that it is of order N smaller and the result follows.

$\square$

The above lemma gives us the following result

Lemma 3.3.11

$$N^{\beta}\zeta_0^N E_N \longrightarrow 0 \qquad as \qquad N \longrightarrow \infty. \tag{3.76}$$

Proof By Lemma 3.3.10

$$N^{\beta}\zeta_0^N E_N \leq 2N^{\beta+2}\zeta_0^N \sum_{i=0}^{\lfloor\frac{N+1}{2}\rfloor-1} w_{i+1}^2[\zeta^{N-1}]\left\{\mathcal{Y}_{i+1}(\zeta)\mathcal{L}_{i+1}(\zeta)^{2i-1}\right\}. \tag{3.77}$$

The sum on the right hand side has the same form as $Z_{1,N}$ with β replaced by 2β, cf. Equation (3.47). Equation (3.55), which describes the asymptotic behaviour of $Z_{1,N}$, can therefore be applied to show that the right hand side is $o(N^{2-\beta})$. Since $\beta > 2$, this converges to zero as $N \longrightarrow \infty$.

$\square$

Combining Lemmas 3.3.8 and 3.3.11 completes the proof of Theorem 3.3.1.

3.3.3 Generalization of Z_N

For technical reasons which are relevant in the next section, we need to generalize the sequence Z_N in the following way. In a tree τ, denote the unique nearest neighbour to the root r by s. Define

$$Z_N^{(R)} = \sum_{\tau\in\Gamma_N} w_{\sigma(s)+R-1} \prod_{i\in V(\tau)\setminus\{r,s\}} w_{\sigma(i)}. \tag{3.78}$$

In analogy with (3.9) and (3.10), define the generating functions

$$\mathcal{Z}(\zeta, R) = \sum_{N=1}^{\infty} Z_N^{(R)} \zeta^N \tag{3.79}$$

and

$$g_R(z) = \sum_{n=0}^{\infty} w_{n+R} z^n. \tag{3.80}$$

Clearly $Z_N = Z_N^{(1)}, \mathcal{Z}(\zeta) = \mathcal{Z}(\zeta, 1)$ and $g(z) = g_1(z)$. With the same arguments as for (3.11) we find the relation

$$\mathcal{Z}(\zeta, R) = \zeta g_R(\mathcal{Z}(\zeta)). \tag{3.81}$$

Let $\mathcal{Z}_{0,R} = \mathcal{Z}(\zeta_0, R)$. The following lemma is a generalization of Theorem 3.3.1.

Lemma 3.3.12 *For the branching weights (3.22) which satisfy (3.23) it holds that*

$$Z_N^{(R)} = \left(1 - m + \frac{g_R'(1)}{g(1)}\right)(1-m)^{-\beta} N^{-\beta} \zeta_0^{1-N} \left(1 + o(1)\right). \tag{3.82}$$

Proof We write

$$Z_N^{(R)} = Z_{1,N}^{(R)} + E_N^{(R)} \tag{3.83}$$

in analogy with (3.26). One can show with the same methods as in the previous subsection that $\lim_{N\to\infty} E_N^{(R)}/Z_N = 0$. Therefore we focus on the term $Z_{1,N}^{(R)}$, the contribution from trees for which there is exactly one vertex of maximum degree. We split this term into the case when the maximum degree vertex is the next neighbour of the root and when it is not. We can then write

$$Z_{1,N}^{(R)} = \sum_{i=0}^{N-1} w_{i+R}[\zeta^N]\left\{\zeta \mathcal{L}_i(\zeta)^i\right\} + \sum_{i=0}^{N-2} w_{i+1}[\zeta^N]\left\{\zeta \ell_{i,R}'(\mathcal{L}_i(\zeta))\mathcal{Y}_i(\zeta)\mathcal{L}_i(\zeta)^i\right\} \tag{3.84}$$

where we defined

$$\ell_{i,R}(z) = \sum_{n=0}^{i-1} w_{n+R} z^n. \tag{3.85}$$

Let

$$h(z) = \frac{z^{i+1}}{\ell_i(z)} \tag{3.86}$$

and

$$k(z) = \frac{\ell'_{i,R}(z) z^{i+2}}{\ell_i(z)\left(\ell_i(z) - z\ell'_i(z)\right)}. \tag{3.87}$$

Using the Lagrange inversion formula for the functions h and k we find that

$$[\zeta^N]\left\{\zeta \mathcal{L}_i(\zeta)^i\right\} = \frac{1}{N}[z^{N-i-1}]\left\{\left(\frac{i+1}{\ell_i(z)} - \frac{z\ell'_i(z)}{\ell_i(z)^2}\right)\ell_i(z)^N\right\} \tag{3.88}$$

and

$$\begin{aligned}[\zeta^N]\left\{\zeta \ell'_{i,R}(\mathcal{L}_i(\zeta))\mathcal{Y}_i(\zeta)\mathcal{L}_i(\zeta)^i\right\} &= \frac{1}{N}[z^{N-i-2}]\Bigg\{\left(\frac{(i+2)\ell'_{i,R}(z)}{\ell_i(z)(\ell_i(z) - z\ell'_i(z))}\right. \\ &\quad \left. + z\frac{d}{dz}\left(\frac{\ell'_{i,R}(z)}{\ell_i(z)(\ell_i(z) - z\ell'_i(z))}\right)\right)\ell_i(z)^N\Bigg\}.\end{aligned} \tag{3.89}$$

We now use exactly the same arguments as in the proof of Lemma 3.3.8 to evaluate the asymptotic behaviour of (3.84). One can show that the contribution from the second term in the curly brackets in (3.88) and (3.89) is negligible. Then one can show that for any $\epsilon > 0$

$$\liminf_{N\to\infty} N^\beta \zeta_0^{N-1} Z_{1,N}^{(R)} \geq (1-\epsilon)\left(1-(m-\epsilon)\right)^{1-\beta}\left(1 + \frac{g'_R(1)}{g(1) - g'(1)}\right)$$

and

$$\limsup_{N\to\infty} N^\beta \zeta_0^{N-1} Z_{1,N}^{(R)} \leq (1+\epsilon)\left(1-(m+\epsilon)\right)^{1-\beta}\left(1 + \frac{g'_R(1)}{g(1) - g'(1)}\right).$$

Since this holds for all $\epsilon > 0$ the result follows.

□

3.4 Properties of the finite volume measures

In this section we study some properties of the measures ν_N for the three different scenarios discussed in Section 3.2. We let m denote the mean offspring probability defined in (3.17). The three cases are the generic, critical case ($w_1 < w_c$, $m = 1$), the nongeneric, critical case ($w_1 = w_c$, $m = 1$) and the nongeneric, subcritical case ($w_1 > w_c$, $m < 1$).

All results stated for generic trees are already known [38] but are rederived here in a slightly different way. In the generic case, Equation (3.11) can be solved for $\mathcal{Z}(\zeta)$ close to the critical point ζ_0 and one can then find the asymptotic behaviour of Z_N, the coefficients of $\mathcal{Z}(\zeta)$, see [58, Theorem 3.1] [1]. In the nongeneric critical case, the function $Z(\zeta)$ has the same critical behaviour as in the generic case as long as $g''(1) < \infty$, see [50, Lemma A.2]. With the same arguments as in [42,50] one gets the following result for $Z_N^{(R)}$.

Lemma 3.4.1 *Under the stated assumption on the branching weights (3.6) and assuming that* $m = 1$ *and* $g''(\mathcal{Z}_0) < \infty$ *it holds that*

$$Z_N^{(R)} = \sqrt{\frac{g(\mathcal{Z}_0)}{2\pi g''(\mathcal{Z}_0)}}\zeta_0 g_R'(\mathcal{Z}_0) N^{-3/2}\zeta_0^{-N}\left(1 + o(1)\right). \tag{3.90}$$

In particular [50, 58]

$$Z_N = \sqrt{\frac{g(\mathcal{Z}_0)}{2\pi g''(\mathcal{Z}_0)}} N^{-3/2}\zeta_0^{-N}\left(1 + o(1)\right). \tag{3.91}$$

An analogous result for the asymptotic behaviour of Z_N, for a special choice of branching weights corresponding to nongeneric, critical trees with $g''(1) = \infty$, is stated in [42, VI.18 and VI.19, page 407]. A generalization to $Z_N^{(R)}$ is straightforward and is given in the next lemma.

Lemma 3.4.2 *For the nongeneric, critical branching weights defined by (3.22), with* $2 < \beta < 3$ *and* $w_1 = w_c$ *the following holds*

$$Z_N^{(R)} = C\zeta_0 g_R'(1) N^{-\frac{\beta}{\beta-1}}\zeta_0^{-N}\left(1 + o(1)\right) \tag{3.92}$$

where $C > 0$ *is a constant.*

We now prove that the measures ν_N converge for a certain type of asymptotic behaviour of $Z_N^{(R)}$ and characterize their limit.

Theorem 3.4.3 *If*

$$Z_N^{(R)} = C\left(1 - m + \zeta_0 g_R'(\mathcal{Z}_0)\right) N^{-\delta}\zeta_0^{-N}(1 + o(1)) \tag{3.93}$$

---

[1] See also [42, Theorem VI.6, page 404].

where C *is a positive constant and* $\delta > 1$*, then the measures* ν_N *converge weakly, as* $N \longrightarrow \infty$*, to a probability measure* ν *which has the following properties:*

- *If* $m = 1$*,* ν *is concentrated on the set of trees with exactly one infinite spine having finite, independent, critical Galton–Watson outgrowths defined by the offspring probabilities in (3.15). The numbers* i *and* j *of left and right outgrowths from a vertex on the spine are independently distributed by*

$$\phi(i,j) = \frac{1}{m}\zeta_0 w_{i+j+2}\mathcal{Z}_0^{i+j}. \tag{3.94}$$

- *If* $m < 1$*,* ν *is concentrated on the set of trees with exactly one vertex of infinite degree which we denote by* t*. The length* ℓ *of the path* (r,t) *is distributed by*

$$\psi(\ell) = (1-m)m^{\ell-1}. \tag{3.95}$$

  The outgrowths from the path (r,t) *are finite, independent, subcritical Galton–Watson trees defined by the offspring probabilities in (3.15). The numbers* i *and* j *of left and right outgrowths from a vertex* $v \in (r,t), v \neq t$ *are independently distributed by (3.94).*

Proof First we prove existence of ν. Since the metric space (Γ, d) has the properties stated in Propositions (3.1.2–3.1.3) it is enough, as was explained in Section 1.3.2, to show that for any $k \in \mathbb{N}$ and $\tau' \in \Gamma'$ the probabilities

$$\nu_N\left(\mathcal{B}_{\frac{1}{k}}(\tau')\right) \tag{3.96}$$

converge as $N \longrightarrow \infty$. The ball in (3.96) can be written as

$$\mathcal{B}_{\frac{1}{k}}(\tau') = \{\tau \in \Gamma \mid L_R(\tau) = \tau_0\} \tag{3.97}$$

for some R where $\tau_0 = L_R(\tau')$. Denote the number of vertices in τ_0 of degree R by S and the number of vertices in τ_0 at distance R from the root by T. It is clear that $S + T \geq 0$.

We can now write

$$\nu_N\left(\{\tau \in \Gamma \mid L_R(\tau) = \tau_0\}\right) = \\ Z_N^{-1} W_0 \sum_{N_1+\ldots+N_{S+T}=N-|\tau_0|+T+S} \prod_{i=1}^{S} Z_{N_i}^{(R)} \prod_{j=S+1}^{S+T} Z_{N_j} \tag{3.98}$$

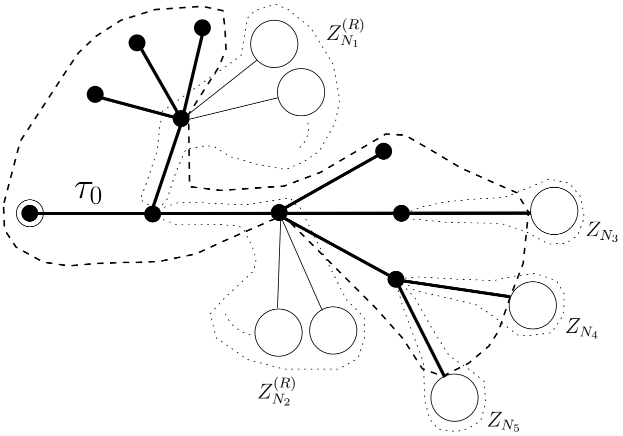


Figure 3.9: An example of the set (3.97) where $R = 4$, $S = 2$ and $T = 3$. When conditioning on trees of size N one attaches the weights $Z^{(R)}_{N_i}$, $i = 1 \ldots S$ and Z_{N_j}, $j = S+1 \ldots S+T$ as indicated in the figure.

where

$$W_0 = \prod_{\substack{v\in V(\tau_0)\setminus\{r\}\\ \sigma(v),|(r,v)|\neq R}} w_{\sigma(v)} \tag{3.99}$$

is the weight of the tree τ_0 (apart from the vertices which are explicitly excluded), and $|(r,v)|$ denotes the length of the path (r,v), see Fig. 3.9. For one of the indices k in each term of the above sum it holds that $N_k \geq \frac{N-|\tau_0|+S+T}{S+T}$. Consider the contribution from terms for which $N_n > A$ for some other index $n \neq k$ and $A > 0$. The indices n and k can belong to either one of the sets $\{1,\ldots,S\}$ or $\{S+1,\ldots,S+T\}$, in total four possibilities. First assume that $S \geq 2$ and $n,k \in \{1,\ldots,S\}$. Using (3.93), this contribution can be estimated from above by

$$\begin{aligned}
&C_1\zeta_0^N Z_N S^2 \sum_{\substack{N_1+\ldots+N_{S+T}=N-|\tau_0|+T+S\\ N_1\geq \frac{N-|\tau_0|+S+T}{S+T},\ N_2>A}} Z^{(R)}_{N_1}\zeta_0^{N_1}\prod_{i=2}^{S} Z^{(R)}_{N_i}\zeta_0^{N_i}\prod_{j=S+1}^{S+T} Z_{N_j}\zeta_0^{N_j}\\
&\leq C_2\left(\frac{(S+T)N}{N-|\tau_0|+T+S}\right)^{\delta} \sum_{\substack{N_3,\ldots,N_{S+T}\geq 1\\ N_2>A}} \prod_{i=2}^{S} Z^{(R)}_{N_i}\zeta_0^{N_i}\prod_{j=S+1}^{S+T} Z_{N_j}\zeta_0^{N_j}\\
&\leq C_3\mathcal{Z}_{0,R}^{S-2}\mathcal{Z}_0^{T}\sum_{N_2>A} N_2^{-\delta} \leq C_4 A^{1-\delta}
\end{aligned}$$

where C_1, C_2, C_3 and C_4 are positive numbers independent of N and A. Exactly the same upper bound is obtained, up to a constant, for the other possible values of k and n. The last expression goes to zero as $A \longrightarrow \infty$ since $\delta > 1$. The remaining contribution to the probability (3.98) is then

$$\begin{aligned}
&\sum_{k=1}^{S+T} Z_N^{-1} W_0 \sum_{\substack{N_1+\ldots+N_{S+T}=N-|\tau_0|+T+S\\ N_n\leq A,\ n\neq k}} \prod_{i=1}^{S} Z^{(R)}_{N_i} \prod_{j=S+1}^{S+T} Z_{N_j}\\
&\xrightarrow[N\to\infty]{} W_0\zeta_0^{|\tau_0|-S-T}\Bigg(S(1-m+\zeta_0 g_R'(Z_0))\left(\sum_{n=1}^{A} Z_n^{(R)}\zeta_0^n\right)^{S-1}\left(\sum_{n=1}^{A} Z_n\zeta_0^n\right)^{T}\\
&\qquad\qquad + T\left(\sum_{n=1}^{A} Z_n^{(R)}\zeta_0^n\right)^{S}\left(\sum_{n=1}^{A} Z_n\zeta_0^n\right)^{T-1}\Bigg)\\
&\xrightarrow[A\to\infty]{} W_0\zeta_0^{|\tau_0|-S-T}\left(S(1-m+\zeta_0 g_R'(\mathcal{Z}_0))\mathcal{Z}_{0,R}^{S-1}\mathcal{Z}_0^{T}+T\mathcal{Z}_{0,R}^{S}\mathcal{Z}_0^{T-1}\right).
\end{aligned}\tag{3.100}$$

This completes the proof of the existence of ν. We now characterize ν separately for the cases $m = 1$ and $m < 1$.

<u>$m = 1$:</u> Let A_R be the set of all trees which have a path (r, s_R) of length R, exactly one possibly infinite tree attached to s_R and all other trees attached to (r, s_R) finite, see Fig. 3.10. Using (3.100) one finds that

$$\nu(A_R) = 1 \tag{3.101}$$

for all R and therefore by taking R to infinity one finds that ν is concentrated on trees with exactly one spine having finite outgrowths. The distribution of the outgrowths follows from (3.100).

<u>$m < 1$:</u> Let $A_{R,\ell}$ be the set of all trees which have a path (r, t) of length ℓ where $\sigma(t) \geq R$. Furthermore, the trees attached to t in the the R–th, $R+1$–st, … position

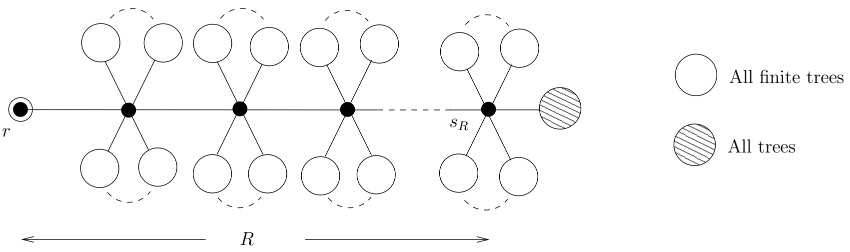


Figure 3.10: An illustration of the set A_R.

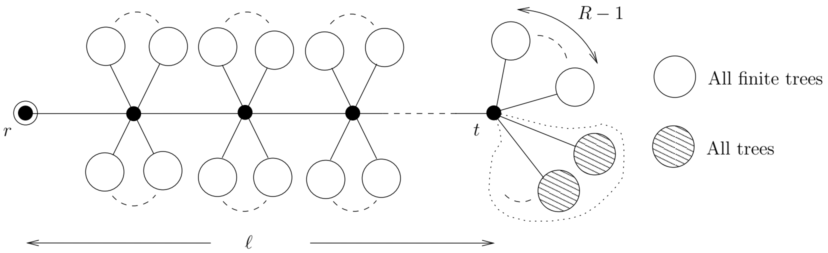


Figure 3.11: An illustration of the set $A_{R,\ell}$.

clockwise from (r,t) are possibly infinite but all other outgrowths from (r,t) are finite, see Fig. 3.11. Using (3.100) one finds that

$$\nu(A_{R,\ell}) = \left(1 - m + \frac{g_R'(1)}{g(1)}\right) m^{\ell-1}. \tag{3.102}$$

The sets $A_{R,\ell}$ are decreasing in R so taking R to infinity in (3.102) one finds, by the monotone convergence theorem, that the probability of exactly one vertex having an infinite degree and being at a distance ℓ from the root is $(1-m)m^{\ell-1}$. Summing this over ℓ gives 1 which shows that the measure is concentrated on trees with exactly one vertex of infinite degree. The distribution of the outgrowths follows from (3.100).

□

Theorem 3.4.4 *Theorem 3.4.3 applies to the generic, critical ensemble in Lemma 3.4.1, the nongeneric, critical ensemble in Lemma 3.4.2 and the nongeneric, subcritical ensembles defined by (3.22) and (3.23).*

Proof This follows from Lemmas 3.3.12, 3.4.1 and 3.4.2 since (3.93) holds with

$$\delta = \begin{cases} 3/2 & \text{generic, and nongeneric critical with } g''(1) < \infty \\ \beta/(\beta-1) & \text{nongeneric critical with } 2 < \beta < 3 \\ \beta & \text{nongeneric subcritical.} \end{cases} \tag{3.103}$$

□

The next result concerns the size of the large vertex, in finite trees, which arises in the nongeneric, subcritical phase.

Theorem 3.4.5 *Consider the nongeneric branching weights defined by (3.22) and (3.23). Let $C_{N,\epsilon}$ be the event that a tree in Γ_N has exactly one vertex of maximum* *degree* $\sigma_{\max}$ *and* $(1-m-\epsilon)N \leq \sigma_{\max} \leq (1-m+\epsilon)N$. *For any* $\epsilon, \delta > 0$ *there exists an* $N_0 \in \mathbb{N}$ *such that*

$$\nu_N\left(C_{N,\epsilon}\right) > 1-\delta \tag{3.104}$$

for all $N \geq N_0$.

Proof This follows directly from the estimates (3.60), (3.62), (3.65–3.70) and (3.76).

□

3.5 The spectral dimension of subcritical trees

It is clear, as in the case of subcritical caterpillars, that the Hausdorff and spectral dimensions of subcritical trees are almost surely infinite. However, it turns out that the annealed spectral dimension is finite and in fact, it takes the same values as in the case of the subcritical caterpillars. The main result of this section is the following theorem.

Theorem 3.5.1 *For any* $\beta > 2$ *the annealed spectral dimension of the subcritical trees defined by (3.22) and (3.23) is*

$$\bar{d}_s = 2(\beta-1). \tag{3.105}$$

We will prove separately a lower bound and an upper bound on $\bar{d}_s$. We first present Faà di Bruno's formula for the n–th derivative of a composite function (see e.g. [12]) which will be used repeatedly.

Lemma 3.5.2 *(Faà di Bruno's formula) If* f *and* g *are* n *times differentiable functions then*

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{\sum_{i=1}^n iq_i = n} \frac{n!}{q_1!q_2!\cdots q_n!} f^{(q_1+\ldots+q_n)}(g(x)) \prod_{j=1}^{n} \left(\frac{g^j(x)}{j!}\right)^{q_j}. \tag{3.106}$$

The following lemma will be needed to obtain the lower bound on $\bar{d}_s$.

Lemma 3.5.3 *Let* μ *be a subcritical Galton–Watson measure on* Γ *corresponding to the offspring probabilities (3.15). For any* $n < \beta - 1$ *and any nonnegative integers* $\theta_1, \ldots, \theta_k$, $k \leq n$ *such that* $\theta_k \neq 0$ *and* $\sum_{a=1}^k a\theta_a \leq n$ *it holds that*

$$\left\langle \prod_{a=1}^{k} \left((-1)^a P_T^{(a)}(x)\right)^{\theta_a} \right\rangle_\mu < \infty \tag{3.107}$$

for all $x \in [0,1]$.

Proof The result is obvious for $x > 0$ since the coefficients of $P_T(x)$ are smaller than one. First, take a fixed finite tree T with root of degree one. Denote the degree of the nearest neighbour of the root by N and the finite trees attached to that vertex by $T_1, \ldots, T_{N-1}$. Then from [38] we have the recursion

$$P_T(x) = \frac{1-x}{S_T(x)} \tag{3.108}$$

where

$$S_T(x) = N - \sum_{i=1}^{N-1} P_{T_i}(x). \tag{3.109}$$

Note that $S_T(x) \geq 1$, since $P_{T_i}(x) \leq 1$ for all i. By Faà di Bruno's formula (with $f(x) = 1/x$, $g(x) = S_T(x)$) and using $S_T(x) \geq 1$ we find that

$$\begin{aligned} \frac{(-1)^b P_T^{(b)}(x)}{b!} \quad \leq & \sum_{\sum_{i=1}^b iq_i = b} \binom{q_1 + \cdots + q_b}{q_1, \ldots, q_b} \prod_{j=1}^{b} \left(\frac{(-1)^{j+1} S_T^{(j)}(x)}{j!} \right)^{q_j} \\ & + \sum_{\sum_{i=1}^{b-1} iq_i = b-1} \binom{q_1 + \cdots + q_{b-1}}{q_1, \ldots, q_{b-1}} \prod_{j=1}^{b-1} \left(\frac{(-1)^{j+1} S_T^{(j)}(x)}{j!} \right)^{q_j} \end{aligned} \tag{3.110}$$

where $\binom{q_1+\cdots+q_b}{q_1,\ldots,q_b}$ is the multinomial coefficient. Looking at the product from the first sum we find that

$$\prod_{j=1}^{b} \left(\frac{(-1)^{j+1} S_T^{(j)}(x)}{j!} \right)^{q_j} = \prod_{j=1}^{b} \sum_{p_1+\cdots+p_{N-1}=q_j} \binom{q_j}{p_1, \ldots, p_{N-1}} \prod_{i=1}^{N-1} \left(\frac{(-1)^j P_{T_i}^{(j)}(x)}{j!} \right)^{p_i}. \tag{3.111}$$

Expanding the above products and keeping track of the factors in each term which depend on the same outgrowth T_i, $i = 1, \ldots, N-1$ we find that they are of the form

$$C_i \prod_{j=1}^{b} \left(\frac{(-1)^j P_{T_i}^{(j)}(x)}{j!} \right)^{\alpha_j} \tag{3.112}$$

where $\sum_{j=1}^b j\alpha_j \leq b$ and C_i is a number independent of T_i (the terms in the latter sum in (3.110) are of the same form, if b is replaced by $b-1$). The equality $\sum_{j=1}^b j\alpha_j = b$ holds only when $p_i = \alpha_j = q_j$ in which case $p_a = 0$ if $a \neq i$ and $C_i = 1$. The total contribution from such terms in (3.111) is therefore

$$\sum_{i=1}^{N-1}\prod_{j=1}^{b}\left(\frac{(-1)^j P_{T_i}^{(j)}(x)}{j!}\right)^{q_j}. \tag{3.113}$$

Now choose numbers $\theta_1, \ldots, \theta_k$ such that $\theta_k \neq 0$ and $\sum_{a=1}^{k} a\theta_a \leq n$. Define $\Theta = \sum_{a=1}^{k} a\theta_a$. The following product of (3.110) over b has an upper bound

$$\prod_{b=1}^{k}\left(\frac{(-1)^b P_T^{(b)}(x)}{b!}\right)^{\theta_b} \leq \sum_{i=1}^{N-1}\prod_{b=1}^{k}\left(\frac{(-1)^j P_{T_i}^{(b)}(x)}{b!}\right)^{\theta_b}$$
$$+C\sum_{M=1}^{\Theta}\sum_{\alpha(M)}\sum_{1\leq i_1<i_2<\cdots<i_M\leq N-1}\prod_{p=1}^{M}\prod_{b=1}^{k}\left(\frac{(-1)^b P_{T_{i_p}}^{(b)}(x)}{b!}\right)^{\alpha_{b,i_p}} + C$$

where $\sum_{\alpha(M)}$ is a sum over nonnegative integers α_{b,i_p} which satisfy either

$$\text{(i)} \sum_{b=1}^{k} b\alpha_{b,i_p} < \Theta \qquad \text{or} \qquad \text{(ii)} \sum_{b=1}^{k-1} b\alpha_{b,i_p} = \Theta \tag{3.114}$$

and C is a number which only depends on k and $(\theta_1, \ldots, \theta_k)$. Taking the μ expectation value of the above inequality and using the fact that the subtrees T_i, $i = 1, \ldots, N-1$ are identically and independently distributed and distributed as T itself, yields

$$\left\langle\prod_{b=1}^{k}\left(\frac{(-1)^b P_T^{(b)}(x)}{b!}\right)^{\theta_b}\right\rangle_\mu \leq m\left\langle\prod_{b=1}^{k}\left(\frac{(-1)^b P_T^{(b)}(x)}{b!}\right)^{\theta_b}\right\rangle_\mu$$
$$+\frac{C}{g(1)}\sum_{M=1}^{\Theta}\sum_{\alpha(M)}\frac{g^{(M)}(1)}{M!}\prod_{p=1}^{M}\left\langle\prod_{b=1}^{k}\left(\frac{(-1)^b P_{T_{i_p}}^{(b)}(x)}{b!}\right)^{\alpha_{b,p}}\right\rangle_\mu + C$$

and thus

$$\left\langle\prod_{b=1}^{k}\left(\frac{(-1)^b P_T^{(b)}(x)}{b!}\right)^{\theta_b}\right\rangle_\mu \leq$$
$$\frac{C}{(1-m)g(1)}\sum_{M=1}^{\Theta}\sum_{\alpha(M)}\frac{g^{(M)}(1)}{M!}\prod_{p=1}^{M}\left\langle\prod_{b=1}^{k}\left(\frac{(-1)^b P_{T_{i_p}}^{(b)}(x)}{b!}\right)^{\alpha_{b,p}}\right\rangle_\mu + \frac{C}{1-m}. \tag{3.115}$$

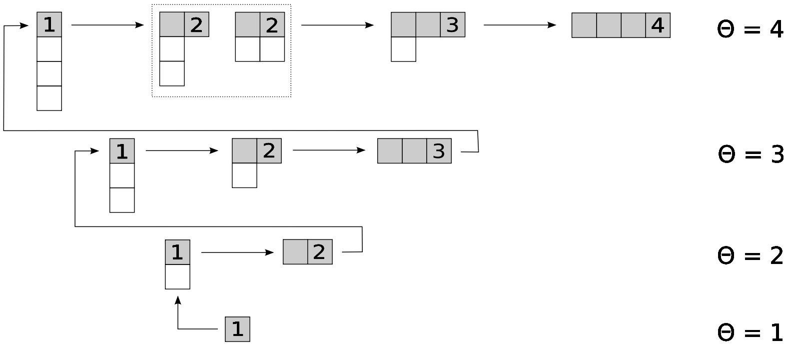


Figure 3.12: A sequence $(\theta_1, \theta_2, \ldots, \theta_k)$ is represented by a Young tableau where θ_i represents the number of rows of size i. The size of a tableau is Θ and the number of elements in the top row (grey boxes) is the value of k. The tableaux are first ordered by Θ and then by k if possible. Tableaux with the same values of Θ and k are incomparable.

Note, that $M \leq \Theta \leq n < \beta - 1$ and thus $g^{(M)}(1) < \infty$. Therefore, for $x > 0$, the right hand side of (3.115) is finite. To show that the left hand side is finite at $x = 0$ we proceed by induction on the sequences $(\theta_1, \theta_2, \ldots, \theta_k)$. We define a partial ordering on the set of such sequences in the following way (see also Fig. 3.12). Sequences $(\theta_1, \ldots, \theta_k)$ and $(\theta'_1, \ldots, \theta'_\ell)$ obey $(\theta'_1, \ldots, \theta'_\ell) < (\theta_1, \ldots, \theta_k)$ if and only if

$$\text{(i)} \sum_{i=1}^{\ell} i\theta'_i < \sum_{i=1}^{k} i\theta_i \qquad \text{or} \qquad \text{(ii)} \sum_{i=1}^{\ell} i\theta'_i = \sum_{i=1}^{k} i\theta_i \quad \text{and} \quad \ell < k.$$

For the smallest values, $k = 1$ and $\Theta = 1$, we find with the same calculations as above that

$$\langle -P'_T(x) \rangle_\mu \leq \frac{1}{1-m}. \tag{3.116}$$

Next assume that (3.107) holds for for all sequences $(\theta'_1, \theta'_2, \ldots, \theta'_{k'})$ which are less than a given sequence $(\theta_1, \theta_2, \ldots, \theta_k)$ with $k, \Theta \leq n$. Then, by (3.114), all the terms on the right hand side of (3.115) are finite and therefore the left hand side is finite for all $x \in [0, 1]$. This shows that (3.107) holds for the sequence $(\theta_1, \theta_2, \ldots, \theta_k)$.

□

3.5.1 A lower bound on $\bar{d}_s$

To find a lower bound on $\bar{d}_s$ we study an upper bound on a suitable derivative of the average return probability generating function. Let M_ℓ be a linear graph of length ℓ

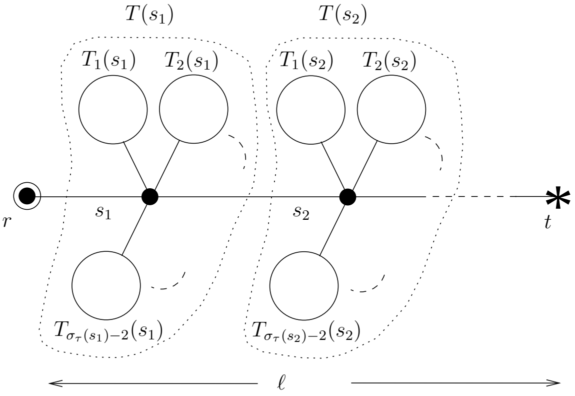


Figure 3.13: A tree from $B_{\ell,k}$.

with the root at one end and a vertex of infinite degree (trap) on the other end. Let $B_{\ell,k}$ be the set of trees with distance ℓ between root and trap and such that at least one vertex on the spine has degree k and all the other vertices have degree no greater than k, cf. proof of Theorem 2.4.4 in Section 2.4. We can write

$$\langle Q_\tau(x)\rangle_\nu = \sum_{\ell=1}^{\infty}\psi(\ell)\sum_{k=2}^{\infty}c(k,\ell)\sum_{\tau\in B_{\ell,k}}\nu(\tau\mid\tau\in B_{\ell,k})Q_\tau(x) \tag{3.117}$$

where

$$c(k,\ell) = \left(\sum_{i+j\le k-2}\phi(i,j)\right)^{\ell-1} - \left(\sum_{i+j\le k-3}\phi(i,j)\right)^{\ell-1}. \tag{3.118}$$

In a tree in $B_{\ell,k}$, denote the root by r, the trap by t and the vertices on the spine by $s_1, s_2, \ldots, s_{\ell-1}$. Denote the outgrowths attached to s_i by $T(s_i)$, where $i = 1, \ldots, \ell-1$ and denote the j-th outgrowth from s_i by $T_j(s_i)$ where $j = 1, \ldots, \sigma_\tau(s_i) - 2$, see Fig. 3.13. The first return probability generating function for $T(s_i)$ (viewing s_i as the root) can be written in terms of the first return probability generating functions for $T_j(s_i)$ in the following way

$$P_{T(s_i)}(x) = \frac{1}{\sigma_\tau(s_i)-2}\sum_{j=1}^{\sigma_\tau(s_i)-2}P_{T_j(s_i)}(x). \tag{3.119}$$

Now take a $\tau \in B_{\ell,k}$. We can write

$$Q_\tau(x) = \sum_{\substack{\omega:\ r\to r \\ \text{on } M_\ell}} K_\tau(x,\omega) W_{M_\ell}(\omega)(1-x)^{|\omega|/2} \tag{3.120}$$

where

$$K_\tau(x,\omega) = \prod_{\substack{t=1 \\ \omega_t \in \{s_1,\ldots,s_{\ell-1}\}}}^{|\omega|-1} \frac{2}{2+(\sigma_\tau(\omega_t)-2)(1-P_{T(\omega_t)}(x))} \tag{3.121}$$

and

$$W_{M_\ell}(\omega) = \prod_{t=0}^{|\omega|-1} (\sigma_{M_\ell}(\omega_t))^{-1}. \tag{3.122}$$

Choose n such that $n+1 < \beta \leq n+2$. Differentiating n times we get

$$\frac{(-1)^n Q_\tau^{(n)}(x)}{n!} = \sum_{n_1+n_2=n} \sum_{\substack{\omega:\ r\to r \\ \text{on } M_\ell}} W_{M_\ell}(\omega) \frac{(-1)^{n_1} K_\tau^{(n_1)}(x,\omega)}{n_1!} \frac{(-1)^{n_2}}{n_2!} \frac{d^{n_2}}{dx^{n_2}} (1-x)^{|\omega|/2}. \tag{3.123}$$

Let ω be a random walk and denote the subwalk of ω which only travels on the vertices $s_1,\ldots,s_{\ell-1}$ by ω'. Denote the number of vertices in ω' by $|\omega'|$ and the t–th vertex in ω' by ω'_t. Then

$$\frac{(-1)^m K_\tau^{(m)}(x,\omega)}{m!} = \sum_{n_1+\cdots+n_{|\omega'|}=m} \prod_{t=1}^{|\omega'|} \frac{(-1)^{n_t}}{n_t!} \frac{d^{n_t}}{dx^{n_t}} \left(\frac{2}{2+(\sigma_\tau(\omega'_t)-2)(1-P_{T(\omega'_t)}(x))} \right).$$

By Faà di Bruno's formula we get

$$\begin{aligned}
&\frac{(-1)^p}{p!} \frac{d^p}{dx^p} \left(\frac{2}{2+(\sigma_\tau(\omega'_t)-2)(1-P_{T(\omega'_t)}(x))} \right) = \\
&\frac{2}{2+(\sigma_\tau(\omega'_t)-2)(1-P_{T(\omega'_t)}(x))} \sum_{q_1+2q_2+\cdots+pq_p=p} \binom{q_1+\cdots+q_p}{q_1,\ldots,q_p} \\
&\times \underbrace{\left(\frac{2(\sigma_\tau(\omega'_t)-2)}{2+(\sigma_\tau(\omega'_t)-2)(1-P_{T(\omega'_t)}(x))} \right)}_{(*)}^{q_1+\cdots+q_p} \prod_{a=1}^{p} \left(\frac{(-1)^a P_{T(\omega'_t)}^{(a)}(x)}{a!} \right)^{q_a}.
\end{aligned}$$

Now, $P_{T(\omega'_t)}(x) \leq 1-x$. Also note that $(*)$ is increasing in $\sigma_\tau(s_i)$ and since $\sigma_\tau(s_i) \leq k$ for $i = 1, \ldots, \ell-1$ we find that

$$(*) \leq \frac{2(k-2)}{2+(k-2)x}. \tag{3.124}$$

Observe that $\frac{2(k-2)}{2+(k-2)x} \leq 1$ for $k = 2,\ 3$ and that $\frac{2(k-2)}{2+(k-2)x} \geq 1$ for $k \geq 4$. Finally, note that $\binom{q_1+\cdots+q_p}{q_1,\ldots,q_p} \leq p^p$. Combining these results and using (3.119) we get the upper bound

$$\begin{aligned}\frac{(-1)^m K_\tau^{(m)}(x,\omega)}{m!} \leq m^m &\left(\frac{2(k-2)}{2+(k-2)x}\right)^{(1-\delta_{k,2})(1-\delta_{k,3})m} \\ &\sum_{n_1+\cdots+n_{|\omega'|}=m} \prod_{t=1}^{|\omega'|} \sum_{q_1+2q_2+\cdots+n_t q_{n_t}=n_t} \prod_{a=1}^{n_t} \frac{1}{(\sigma_\tau(\omega'_t)-2)^{q_a}} \\ &\times \sum_{p_1+\cdots+p_{\sigma_\tau(\omega'_t)-2}=q_a} \binom{q_a}{p_1,\ldots,p_{\sigma_\tau(\omega'_t)-2}} \prod_{j=1}^{\sigma_\tau(\omega'_t)-2} \left(\frac{(-1)^a P^{(a)}_{T_j(\omega'_t)}(x)}{a!}\right)^{p_j}.\end{aligned} \tag{3.125}$$

Expanding the above products and keeping track of the factors in each term which depend on the same outgrowth $T_j(s_i)$, $i = 1, \ldots, \ell-1$, $j = 1, \ldots, \sigma_\tau(s_j)-2$, we find that they are of the form

$$C_{ij} \prod_{a=1}^{n} \left((-1)^a P^{(a)}_{T_j(s_i)}(x)\right)^{\theta_a} \tag{3.126}$$

where $\sum_{a=1}^{n} a\theta_a \leq n$ and C_{ij} is independent of $T_j(s_i)$. By Lemma 3.5.3, the expected value of (3.126) over the outgrowths $T_j(s_i)$ is finite, and since the total number of terms in (3.125) is a polynomial in $|\omega'|$ we find that

$$\left\langle (-1)^m K_\tau^{(m)}(x,\omega) \right\rangle_{\nu,\tau\in B_{\ell,k}} \leq H(|\omega|) \left(\frac{2(k-2)}{2+(k-2)x}\right)^{(1-\delta_{k,2})(1-\delta_{k,3})m} \tag{3.127}$$

where $H(|\omega|)$ is a polynomial with positive coefficients. From this inequality and the fact that $(-1)^i Q^{(i)}_{M_\ell}(0)$ is a polynomial in ℓ of degree $2i+1$, it follows that

$$\langle(-1)^n Q^{(n)}_\tau(x)\rangle_{\nu,\tau\in B_{\ell,k}} \leq \sum_{m=0}^{n} S_m(\ell)\left(\frac{2(k-2)}{2+(k-2)x}\right)^{(1-\delta_{k,2})(1-\delta_{k,3})m} \tag{3.128}$$

where $S_m(\ell)$, $m=0,\dots,n$ are polynomials with positive coefficients. From here we proceed as below Equation (2.97) and find that $\bar{d}_s \geq 2(\beta-1)$.

□

3.5.2 An upper bound on $\bar{d}_s$

To find an upper bound on $\bar{d}_s$ we study a lower bound on a suitable derivative of the average return probability generating function. The aim is to cut off the branches of the finite outgrowths from the spine so that only single leaves are left. We then use monotonicity results from [51] to compare return probability generating functions. As before we choose n such that $n+1<\beta\leq n+2$. We begin by differentiating (3.129) n times and throwing away every term in the sum over ℓ except the $\ell=2$ term

$$\left\langle(-1)^n Q^{(n)}_\tau(x)\right\rangle_\nu \geq (1-m)m\sum_{k=2}^{\infty}\sum_{i+j=k-2}\phi(i,j)\left\langle(-1)^n Q^{(n)}_\tau(x)\right\rangle_{\nu,\tau\in B_{2,k}}. \tag{3.129}$$

Let $M_{2,k}$ be the graph constructed by attaching $k-2$ leaves to the vertex s_1 in M_2, cf. proof of Theorem 2.4.4 in Section 2.4 . Take a tree $\tau\in B_{2,k}$. Denote the nearest neighbours of s_1, excluding r and t, by $u_1,\dots,u_{k-2}$. Denote the finite tree attached to u_i by $U(u_i)$, $i=1,\dots,k-2$, and view u_i as its root, see Fig. 3.14. We can write

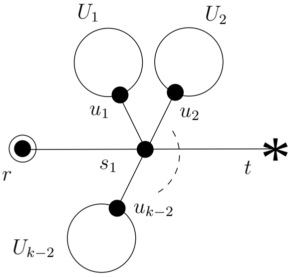


Figure 3.14: A graph $\tau\in B_{2,k}$.

$$Q_\tau(x) = \sum_{\substack{\omega:\ r \to r \\ \text{on } M_{2,k}}} F_\tau(x,\omega) W_{M_{2,k}}(\omega)(1-x)^{|\omega|/2} \tag{3.130}$$

where

$$F_\tau(x,\omega) = \prod_{\substack{t=1 \\ \omega_t \in \{u_1,\ldots,u_{k-2}\}}}^{|\omega|-1} \frac{1}{1+(\sigma_\tau(\omega_t)-1)(1-P_{U(\omega_t)}(x))}. \tag{3.131}$$

Define

$$H(x) = \sum_{\substack{\omega:\ r \to r \\ \text{on } M_{2,k}}} \langle F_\tau(x,\omega)\rangle_{\nu,\tau\in B_{2,k}} W_{M_\ell}(\omega) \frac{d^{n-1}}{dx^n}(1-x)^{|\omega|/2}. \tag{3.132}$$

Differentiating once we easily find that

$$(-1)^n H'(x) \leq \left\langle (-1)^n Q_\tau^{(n)}(x) \right\rangle_{\nu,\tau\in B_{2,k}} \tag{3.133}$$

and using the methods of [51] we find that there exists a sequence ξ_i converging to zero as $i \longrightarrow \infty$ on which

$$(-1)^n Q_{M_{2,k}}^{(n)}(\xi_i) \leq (-1)^n H'(\xi_i). \tag{3.134}$$

Note that from the relation

$$Q_\tau(x) = \frac{1}{1-P_\tau(x)} \tag{3.135}$$

one can show that $(-1)^n Q_\tau(x) \geq (-1)^n P_\tau(x)$ for any τ. Thus, we finally have

$$\left\langle (-1)^n Q_\tau^{(n)}(\xi_i) \right\rangle_\nu \geq (1-m)m \sum_{k=2}^{\infty} \sum_{i+j=k-2} \phi(i,j)(-1)^n P_{M_{2,\ell}}^{(n)}(\xi_i) \tag{3.136}$$

on a sequence ξ_i converging to zero. We now proceed as in Equation (2.91) and find that $\bar{d}_s \leq 2(\beta-1)$.

□

4

Discussion

We have studied an equilibrium statistical mechanical model of two classes of trees: caterpillars and branched polymers. The two classes have identical phase structure, an elongated phase and a condensed phase. We have proven convergence of the Gibbs measures in both phases and on the critical line separating them. The main result is a rigorous proof of the emergence of a vertex of infinite degree in the condensed phase. The phenomenon of condensation seems to appear in more general models of graphs and it would be interesting to prove analogous results in those cases.

In the caterpillar model, we calculated the Hausdorff and spectral dimensions in the generic phase and on the critical line when $g''(1) < \infty$ and found that they are equal to one. In the generic phase of the branched polymer model, it holds that $\bar{d}_H = 2$ and $\bar{d}_s = 4/3$, see [38]. The proof of this result relies only on the fact that the infinite volume measure is concentrated on the set of trees with one infinite spine with finite critical Galton–Watson outgrowths and that $g''(1) < \infty$. Therefore, it follows from Theorem 3.4.3 in the previous chapter that $\bar{d}_H = 2$ and $\bar{d}_s = 4/3$ on the critical line when $g''(1) < \infty$. Note that the equality (1.23) holds in both cases discussed in this paragraph.

We showed that on the critical line in the caterpillar model, when $g''(1) = \infty$, the Hausdorff and spectral dimensions are almost surely

$$d_H = \frac{1}{\beta - 2} \qquad \text{and} \qquad d_s = \frac{2}{\beta - 1} \tag{4.1}$$

with $2 < \beta \leq 3$ where β is the exponent defining the subcritical branching weights

$w_n \sim n^{-\beta}$. The equality (1.23) holds in this case. No analogous results have been proved for the critical line in the branched polymer model when $g''(1) = \infty$. However, scaling arguments suggest that

$$d_H = \frac{\beta - 1}{\beta - 2} \qquad \text{and} \qquad d_s = \frac{2(\beta - 1)}{2\beta - 3} \tag{4.2}$$

where $2 < \beta \leq 3$, see [26,29], and one can check that the equality (1.23) holds. Note that by Theorem 3.4.3, the infinite volume measure is still concentrated on the set of trees with one infinite spine with critical Galton–Watson outgrowths. Therefore, a possible way to prove (4.2) is to follow the arguments in [38], but taking into account the different behaviour of critical Galton–Watson processes having $g''(1) = \infty$. Some results on such Galton–Watson processes can be found in [63].

We have calculated the annealed spectral dimension in the condensed phase in both the caterpillar and branched polymer models, and it takes the values $\bar{d}_s = 2(\beta - 1)$ where $\beta > 2$. This is different from the value $\bar{d}_s = 2$ which was obtained in [29] using scaling arguments. Furthermore, we argued that the annealed Hausdorff dimension is infinite and therefore the inequality (1.22) holds since $2 < \bar{d}_s < \infty$, and $\bar{d}_s$ can take any value in this range.

Part II

Random tree growth by vertex splitting

5

The vertex splitting model

In this part of the thesis we consider a new model of randomly growing trees, referred to as the *vertex splitting model*. We start by defining the model and then we examine some properties of large trees. First, we study the distribution of the degrees of vertices and show that it has a well defined limit as the size of the tree goes to infinity, which is independent of the initial tree. Exact results are provided under certain conditions on the parameters of the model and the general case is supported by simulations.

Secondly, we derive the Hausdorff dimension of the trees by studying the scaling of certain volume distribution functions. We establish bounds on the Hausdorff dimension and show that it can vary continuously with the splitting weights between 1 and $+\infty$. The results we obtain are supported by simulations.

Next, we study the correlations between the degrees of neighbouring vertices. This amounts to studying the density of edges which connect vertices of given degrees. We show that there is a very good agreement between our analytical results and numerical simulations. We conclude by discussing the amount of assortative mixing in the vertex splitting model, i.e. whether vertices of high degree prefer to be neigbours of vertices of high degree or to be neighbours of vertices of low degree.

Finally, we discuss the relationship between our model and other models of random trees, in particular the alpha model of phylogenetic trees. We prove convergence of the finite volume measures generated by the growth rules of the alpha model and calculate the annealed Hausdorff dimension with respect to the infinite volume measure.

5.1 Definition of the model

Let $\Gamma^{(D)}$ be the collection of all rooted planar trees for which every vertex has finite degree at most D. Let $\Gamma_N^{(D)}$ be those trees $T \in \Gamma^{(D)}$ with $|T| = N$. Denote the number of vertices of degree i in T by $n_i(T)$. Let

$$\mathbf{M} = \begin{bmatrix} 0 & w_{1,2} & w_{1,3} & \cdots & w_{1,D-1} & w_{1,D} \\ w_{2,1} & w_{2,2} & w_{2,3} & \cdots & w_{2,D-1} & w_{2,D} \\ w_{3,1} & w_{3,2} & w_{3,3} & \cdots & w_{3,D-1} & 0 \\ w_{4,1} & w_{4,2} & w_{4,3} & & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ w_{D,1} & w_{D,2} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

be a symmetric matrix with nonnegative entries that we call *partitioning weights*. We define a collection of nonnegative numbers called *splitting weights*, $w_1, w_2, \ldots, w_D$, by

$$w_i = \frac{i}{2}\sum_{j=1}^{i+1} w_{j,i+2-j}. \tag{5.1}$$

We now define a growth rule for planar trees which we call *vertex splitting*. Given a tree $T \in \Gamma_N^{(D)}$

(i) Choose a vertex v of T with probability $w_i/\mathcal{W}(T)$ where i is the order of v and

$$\mathcal{W}(T) = \sum_{j=1}^{D} w_j n_j(T). \tag{5.2}$$

(ii) Partition the edges incident with v into two disjoint sets V and V' of adjacent edges with probability

$$\frac{w_{k,i+2-k}}{w_i}.$$

The set V contains $k-1$ of the edges and V' contains $i-(k-1)$ of these edges, $k = 1, \ldots, i$. For a given k, all such partitionings are taken to be equally likely.

(iii) Move all edges in V' from v to a new vertex v' and create an edge joining v to v'. If v is the root, then the new vertex of order one is taken to be the root.

This vertex splitting operation is illustrated in Figure 5.1 (the root vertex is circled).

After the splitting operation, the degree of vertex v is k and the degree of vertex v' is

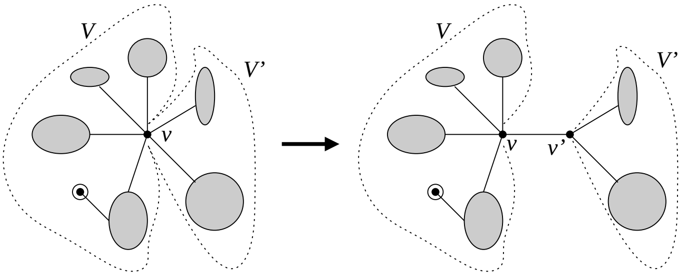


Figure 5.1: Illustration of the splitting process for $i = 6$ and $k = 5$.

$i+2-k$. Since the maximum allowed vertex degree is D we define $w_{D+1,1} = w_{1,D+1} = 0$, i.e. we do not allow splittings of vertices of degree D that produce vertices of degree $D+1$. If the partitioning weights are chosen such that $w_{i,j} = 0$ for $i \neq 1$ or $j \neq 1$, then the vertex splitting model is equivalent to the preferential attachment model discussed in [31].

We will often think of the number of edges as time and denote it by ℓ assuming we start with the single vertex tree at time $\ell = 0$. In Chapters 7 and 8 we will find it convenient to label the vertices according to their time of creation. In this case we append the following to our rules:

(iv) The single root vertex (which is the only tree in $\Gamma_0^{(D)}$) is given the label 0. Let a be the label of the vertex v chosen in (i) at time ℓ. If v is further away from the root than v' in step (iii) then we let v keep the label a and give v' the label $\ell+1$. Otherwise label v with $\ell+1$ and label v' with a.

This book-keeping device has no effect on the dynamics of the model.

If the partitioning weights are chosen such that the splitting weights are linear,

$$w_i = ai + b \tag{5.3}$$

for some a and b, then the model is easier to analyse since the weight of a tree $T \in \Gamma^{(D)}$ depends only on the size of the tree

$$\mathcal{W}(T) = (2a+b)|T| + b. \tag{5.4}$$

This is easily seen from the two constraints on the vertex degrees,

$$\sum_{i=1}^{D} n_i(T) \;=\; |T| + 1 \quad \text{and} \quad \sum_{i=1}^{D} i n_i(T) \;=\; 2|T|. \tag{5.5}$$

By abuse of notation, in this case we will write $\mathcal{W}(|T|) = \mathcal{W}(T)$. We will also sometimes restrict to uniform partitioning weights, i.e.

$$w_{i,k+2-i} = \begin{cases} w_k/\binom{k+1}{2} & \text{for } i = 1, \dots, k+1, \quad \text{if } k < D, \\ w_k/\binom{k}{2} & \text{for } i = 2, \dots, k, \quad \text{if } k = D. \end{cases} \tag{5.6}$$

6

Vertex degree distribution

6.1 The case of linear splitting weights

Start from a finite tree T_0 at time $\ell_0 = |T_0|$ and perform vertex splitting according to the rules described in the previous chapter ℓ_1 times. We then obtain a tree in $\Gamma^{(D)}_{\ell_0+\ell_1}$. Let $\ell = \ell_0 + \ell_1$. The vertex splitting operation induces a probability measure ν_ℓ on $\Gamma^{(D)}_\ell$, which of course depends on the initial tree T_0. In this section we will drop T_0 from function arguments with the understanding that it is implied, unless otherwise stated.

Let $P_\ell(m_1, \ldots, m_D)$ be the probability that $T \in \Gamma^{(D)}_\ell$ has $(n_1(T), \ldots, n_D(T)) = (m_1, \ldots, m_D)$ according to the measure ν_ℓ. We wish to study the mean value of $n_k(T)$ with respect to the measure ν_ℓ. Denote this value by $\overline{n}_{\ell,k}$. We define the vertex degree densities $\rho_{\ell,k} \equiv \overline{n}_{\ell,k}/(\ell+1)$ and with some conditions on the partitioning weights we will prove the existence of the limit

$$\lim_{\ell\to\infty} \rho_{\ell,k} \equiv \rho_k$$

and show that the ρ_k satisfy a system of linear equations.

Let $\mathbf{x} = (x_1, \ldots, x_D) \in \mathbb{R}^D$ and define the probability generating function

$$\mathcal{H}_\ell(\mathbf{x}) \quad = \sum_{n_1+\cdots+n_D=\ell+1} P_\ell(n_1, \ldots, n_D) x_1^{n_1} \cdots x_D^{n_D} \qquad (6.1)$$

Proposition 6.1.1 *The probability generating function* $\mathcal{H}_\ell(\mathbf{x})$ *satisfies the recurrence*

$$\mathcal{H}_{\ell+1}(\mathbf{x}) = \sum_{n_1+\cdots+n_D=\ell+1} \frac{P_\ell(n_1,\ldots,n_D)}{\sum_{i=1}^D n_i w_i} \mathbf{c}(\mathbf{x}) \cdot \nabla(x_1^{n_1}\cdots x_D^{n_D}) \tag{6.2}$$

for all $\ell \geq \ell_0$, *where*

$$\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_D(\mathbf{x})) \tag{6.3}$$

with

$$c_i(\mathbf{x}) = \frac{i}{2} \sum_{j=1}^{i+1} w_{j,i+2-j} x_j x_{i+2-j} \tag{6.4}$$

and $\nabla = \Big(\partial/\partial x_1, \ldots, \partial/\partial x_D\Big)$ *is the standard gradient operator.*

Proof Any tree contributing to $\mathcal{H}_{\ell+1}$ can be obtained by splitting a vertex in a tree with ℓ edges. This process can be divided into three steps:

(i) Choose a tree $T \in \Gamma_\ell^{(D)}$ with vertex degree distribution $(n_1,\ldots,n_D)$ with probability $P_\ell(n_1,\ldots,n_D)$.

(ii) Select a vertex in T of degree i with probability $n_i w_i / \sum_j n_j w_j$.

(iii) Partition the edges incident to the chosen vertex into two sets V and V' of adjacent edges with $j-1$ and $i+1-j$ elements, respectively, with probability $i w_{j,i+2-j}/w_i$ if $j \neq i+2-j$ and with probability $\frac{i}{2} w_{j,i+2-j}/w_i$ if $j = i+2-j$. In the latter case there is a symmetry between V and V' which accounts for the factor $1/2$.

Multiplying together the probabilities in (i)–(iii) gives the probability of removing a vertex of degree i and creating two new vertices of degree j and $i+2-j$. In terms of the generating function this amounts to replacing $x_1^{n_1}\cdots x_D^{n_D}$ by $x_i^{-1} x_j x_{i+2-j} x_1^{n_1}\cdots x_D^{n_D}$. The probability is

$$\frac{P_\ell(n_1,\ldots,n_D)}{\sum_j n_j w_j} n_i \times \begin{cases} i w_{j,i+2-j} & \text{if } j \neq i+2-j, \\ \frac{i}{2} w_{j,i+2-j} & \text{otherwise.} \end{cases}$$

The partial derivative $\partial/\partial x_i$ in ∇ takes care of removing a vertex of degree i and provides the factor n_i. In $c_i(\mathbf{x})$, the factors $x_j x_{i+2-j}$ add two vertices of degree j

and $i+2-j$ respectively and the appropriate weights are given. Now sum over all possible partitionings in (iii), the dot product of $\mathbf{c}(\mathbf{x})$ and ∇ accounts for the sum over all vertex degrees, and finally sum over all vertex degree configurations in the initial tree to obtain (6.2).

□

For linear weights (5.3), Equation (6.2) reduces to a much simpler recursion

$$\mathcal{H}_{\ell+1}(\mathbf{x}) = \frac{1}{\mathcal{W}(\ell)}\mathbf{c}(\mathbf{x}) \cdot \nabla \mathcal{H}_\ell(\mathbf{x}) \tag{6.5}$$

by (5.4), where $\mathcal{W}(\ell) = (2a+b)\ell+b$. The remainder of this subsection concerns linear weights only. We have

$$\overline{n}_{\ell,k} \;=\; \sum_{n_1+\ldots+n_D=\ell+1} P_\ell(n_1,\ldots,n_D)n_k \;=\; \partial_k \mathcal{H}_\ell(\mathbf{x})|_{\mathbf{x}=\mathbf{1}}, \tag{6.6}$$

where $\mathbf{1} = (1,1,\ldots,1)$. To get a recursion equation for $\overline{n}_{\ell,k}$, differentiate both sides of (6.5) with respect to x_k and set $\mathbf{x}=\mathbf{1}$ to find

$$\overline{n}_{\ell+1,k} \;=\; \frac{1}{\mathcal{W}(\ell)}\left(\sum_{i=k-1}^{D} i w_{k,i+2-k}\overline{n}_{\ell,i} + \sum_{i=1}^{D} w_i \partial_i \partial_k \mathcal{H}_\ell(\mathbf{x})|_{\mathbf{x}=\mathbf{1}}\right). \tag{6.7}$$

Since the weights are linear we can use the constraints in (5.5) to rewrite the last term in (6.7) as

$$\sum_{i=1}^{D} w_i \partial_i \partial_k \mathcal{H}_\ell(\mathbf{x})|_{\mathbf{x}=\mathbf{1}} \;=\; \left(-w_k + \mathcal{W}(\ell)\right)\overline{n}_{\ell,k}. \tag{6.8}$$

Inserting this into (6.7) we see that the equations close

$$\overline{n}_{\ell+1,k} \;=\; \frac{1}{\mathcal{W}(\ell)}\left(-w_k\overline{n}_{\ell,k} + \sum_{i=k-1}^{D} i w_{k,i+2-k}\overline{n}_{\ell,i}\right) + \overline{n}_{\ell,k}. \tag{6.9}$$

We can also write the recursion in terms of $\rho_{\ell,k}$ and find

$$(\ell+2)\rho_{\ell+1,k} = \frac{\ell+1}{\mathcal{W}(\ell)}\left(-w_k\rho_{\ell,k} + \sum_{i=k-1}^{D} i w_{k,i+2-k}\rho_{\ell,i}\right) + (\ell+1)\rho_{\ell,k} \tag{6.10}$$

The above equation can be put in the matrix form

$$\boldsymbol{\rho}_{\ell+1} \quad = \quad \mathbf{A}_\ell \, \boldsymbol{\rho}_\ell \tag{6.11}$$

where

$$\boldsymbol{\rho}_\ell \quad = \quad (\rho_{\ell,1}, \rho_{\ell,2}, \ldots, \rho_{\ell,D})^T, \qquad \mathbf{A}_\ell \;=\; \frac{\ell+1}{\ell+2}\left(\mathbf{I} + \frac{1}{\mathcal{W}(\ell)}\mathbf{B}\right), \tag{6.12}$$

$$\mathbf{B} = \begin{bmatrix} w_{1,2} & 2w_{1,3} & \cdots & (D-2)w_{1,D-1} & (D-1)w_{1,D} & 0 \\ w_{2,1} & 2w_{2,2} & \cdots & (D-2)w_{2,D-2} & (D-1)w_{2,D-1} & Dw_{2,D} \\ 0 & 2w_{3,1} & \cdots & (D-2)w_{3,D-3} & (D-1)w_{3,D-2} & Dw_{3,D-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & \ddots & (D-2)w_{D-1,1} & (D-1)w_{D-1,2} & Dw_{D-1,3} \\ 0 & \cdots & 0 & 0 & (D-1)w_{D,1} & Dw_{D,2} \end{bmatrix} - \operatorname{diag}(w_i)_{1\le i\le D} \tag{6.13}$$

and $\mathbf{I}$ is the identity matrix.

If we denote the vertex degree densities of the initial tree T_0 by $\boldsymbol{\rho}_{\ell_0}$ we can write the densities for trees on ℓ edges which grow from the initial tree as

$$\boldsymbol{\rho}_\ell = \left(\prod_{i=\ell_0}^{\ell-1} \mathbf{A}_i\right)\boldsymbol{\rho}_{\ell_0} \quad = \quad \frac{\ell_0+1}{\ell+1}\left(\prod_{i=\ell_0}^{\ell-1}\left(\mathbf{I} + \frac{1}{\mathcal{W}(i)}\mathbf{B}\right)\right)\boldsymbol{\rho}_{\ell_0}. \tag{6.14}$$

We will establish convergence of the right hand side by imposing some technical restrictions on $\mathbf{B}$. It turns out that the limiting distribution is independent of the initial distribution $\boldsymbol{\rho}_{\ell_0}$. We begin with some necessary lemmas.

Lemma 6.1.2 *If* λ *is an eigenvalue of* $\mathbf{B}$ *with corresponding eigenvector* $\mathbf{e}_\lambda = (e_{\lambda 1}, \ldots, e_{\lambda D})$, *i.e.*

$$\mathbf{B}\mathbf{e}_\lambda = \lambda \mathbf{e}_\lambda, \tag{6.15}$$

then the following holds:

$$\lambda \sum_{i=1}^{D} e_{\lambda i} \quad = \quad \sum_{i=1}^{D} w_i e_{\lambda i} \qquad \textit{and} \tag{6.16}$$

$$\lambda \sum_{i=1}^{D} i e_{\lambda i} \quad = \quad 2 \sum_{i=1}^{D} w_i e_{\lambda i}. \tag{6.17}$$

Proof We prove the second identity. The first identity is established by a similar calculation. Multiply the i-th component of the eigenvalue equation (6.15) by i and sum over i to get

$$\begin{aligned} \lambda \sum_{i=1}^{D} i e_{\lambda i} &= -\sum_{i=1}^{D} i w_i e_{\lambda i} + \sum_{i=1}^{D} i \sum_{k=i-1}^{D} k w_{i,k+2-i} e_{\lambda k} \\ &= -\sum_{i=1}^{D} i w_i e_{\lambda i} + \sum_{k=1}^{D} k \left(\sum_{i=1}^{k+1} i w_{i,k+2-i} \right) e_{\lambda k}. \end{aligned} \tag{6.18}$$

Using $w_{i,j} = w_{j,i}$ we find that

$$\sum_{i=1}^{k+1} i w_{i,k+2-i} = \frac{k+2}{2} \sum_{i=1}^{k+1} w_{i,k+2-i} \tag{6.19}$$

and this together with the definition of the splitting weights (5.1) proves the identity.

□

Lemma 6.1.3 *If*

1. $w_{k,1} = w_{1,k} > 0$ *for* $k = 1, \ldots, D$ *(i.e. it is possible to produce vertices of degree* D*) and*

2. $w_{i,D+2-i} > 0$ *for at least one* i *with* $2 \leq i \leq D-1$,

then w_2 *is a positive, simple eigenvalue of* $\mathbf{B}$*. All other eigenvalues of* $\mathbf{B}$ *have a smaller real part. The corresponding eigenvector* $\mathbf{e}_{w_2}$ *can be taken to have all entries positive.*

Proof We begin by choosing a number $\gamma > \max_{1 \leq k \leq D} \{w_k - k w_{k,2}\}$ and define $\mathbf{P} = \mathbf{B} + \gamma \mathbf{I}$. The matrix $\mathbf{P}$ has only nonnegative entries and the conditions (1) and (2) on $\mathbf{B}$ guarantee that it is primitive, i.e. there is a number k such that all entries of the matrix $\mathbf{P}^k$ are positive. Therefore, by the Perron–Frobenius theorem [62], $\mathbf{P}$ has a simple positive eigenvalue r and all other eigenvalues of $\mathbf{P}$ have a smaller modulus. The corresponding eigenvector $\mathbf{e}_r$ can be taken to have all entries positive.

We normalize the eigenvector such that

$$\sum_{i=1}^{D} e_{ri} = 1. \tag{6.20}$$

Shifting back to the matrix $\mathbf{B}$ we find that $w \equiv r - \gamma$ is a simple real eigenvalue of $\mathbf{B}$ with the largest real part and the corresponding eigenvector is $\mathbf{e}_w = \mathbf{e}_r$. We see right away from (6.16) and with the chosen normalization that

$$w = \sum_{i=1}^{D} w_i e_{wi}. \tag{6.21}$$

Since the weights are linear, Lemma 6.1.2 shows that $w = w_2$.

□

Note that the first condition on the weights in the above lemma is natural since we have fixed a maximal degree D and therefore we want to be able to produce vertices of degree D. The second condition, however, does not seem to be necessary for the results to hold but we still require it in order to use the Perron–Frobenius theorem for primitive matrices. This condition is not very restrictive in the case of linear weights since it holds for all a and b except when $aD + b = 0$.

Lemma 6.1.4 *Let $\lambda \in \mathbb{C}$. Then*

$$\frac{\ell_0+1}{\ell+1} \prod_{i=\ell_0}^{\ell-1} \left(1 + \frac{1}{\mathcal{W}(i)}\lambda\right) \longrightarrow \begin{cases} \frac{(\ell_0+1)w_2}{\ell_0 w_2 + b} & \text{if } \lambda = w_2, \\ 0 & \text{if } \mathrm{Re}(\lambda) < w_2 \end{cases} \tag{6.22}$$

as $\ell \longrightarrow \infty$.

Proof The result follows from the identity

$$\frac{\ell_0+1}{\ell+1} \prod_{i=\ell_0}^{\ell-1} \left(1 + \frac{1}{\mathcal{W}(i)}\lambda\right) = \frac{\ell_0+1}{\ell+1} \frac{\Gamma\left(\ell + \frac{b+\lambda}{w_2}\right)\Gamma\left(\ell_0 + \frac{b}{w_2}\right)}{\Gamma\left(\ell + \frac{b}{w_2}\right)\Gamma\left(\ell_0 + \frac{b+\lambda}{w_2}\right)}. \tag{6.23}$$

□

Theorem 6.1.5 *With the assumptions on* $\mathbf{B}$ *in Lemma 6.1.3 and the additional assumption that* $\mathbf{B}$ *is diagonalizable, the limit as* $\ell \longrightarrow \infty$ *of the right hand side of Equation (6.14) exists and is given by the eigenvector* $\mathbf{e}_{w_2}$ *of* $\mathbf{B}$ *normalized such that*

$$\sum_{i=1}^{D} e_{w_2 i} = 1. \tag{6.24}$$

Proof We use the normalization in (6.24) and expand $\boldsymbol{\rho}_{\ell_0}$ in the basis of eigenvectors of $\mathbf{B}$. Using the results of Lemmas 6.1.2 and 6.1.3 and that T_0 satisfies the equations in (5.5) we see that the expansion is of the form

$$\boldsymbol{\rho}_{\ell_0} = \frac{w_2 \ell_0 + b}{w_2(\ell_0 + 1)} \mathbf{e}_{w_2} + \sum_{i=1}^{D-1} a_i \mathbf{e}_{\lambda_i} \tag{6.25}$$

where $\lambda_i, \; i = 1, \ldots, D-1$ are the eigenvalues of $\mathbf{B}$ with real part less than w_2. The result now follows from Lemma 6.1.4.

□

Theorem (6.1.5) shows that with the above conditions on $\mathbf{B}$ the limit of the vertex degree densities exists, is independent of the initial tree and is given by

$$\boldsymbol{\rho} \equiv \lim_{\ell \to \infty} \boldsymbol{\rho}_\ell = \mathbf{e}_{w_2}. \tag{6.26}$$

The limiting densities are therefore the unique positive solution to Equation (6.15), i.e.

$$\rho_k = -\frac{w_k}{w_2} \rho_k + \sum_{i=k-1}^{D} i \frac{w_{k,i+2-k}}{w_2} \rho_i. \tag{6.27}$$

6.2 Explicit solutions

We discuss three simple special cases.

1) When $D = 3$ we find that

$$\mathbf{B} = \begin{bmatrix} 0 & 2w_{1,3} & 0 \\ w_{2,1} & w_{2,2} - 2w_{3,1} & 3w_{3,2} \\ 0 & 2w_{3,1} & 0 \end{bmatrix}. \tag{6.28}$$

If the weights satisfy the conditions in Lemma 6.1.3 it is easy to see that $\mathbf{B}$ is diagonalizable. For linear splitting weights $w_i = ai + b$ and uniform partitioning weights the positive solution of (6.27) is

$$\rho_1 = \rho_3 = \frac{2}{7} \quad \text{and} \quad \rho_2 = \frac{3}{7} \tag{6.29}$$

for all values of a and b as can easily be seen from the simple structure of $\mathbf{B}$ in this case.

2) When $D = 4$, the splitting weights linear and the partitioning weights uniform one can check that

$$\mathbf{B} = \begin{bmatrix} 0 & \frac{2}{3}(2a+b) & \frac{1}{2}(3a+b) & 0 \\ a+b & -\frac{1}{3}(2a+b) & \frac{1}{2}(3a+b) & \frac{2}{3}(4a+b) \\ 0 & \frac{2}{3}(2a+b) & -\frac{1}{2}(3a+b) & \frac{2}{3}(4a+b) \\ 0 & 0 & \frac{1}{2}(3a+b) & -\frac{1}{3}(4a+b) \end{bmatrix}. \tag{6.30}$$

When $4a + b > 0$ the weights satisfy the conditions in Lemma 6.1.3. The eigenvalues of $\mathbf{B}$ are $-\frac{1}{12}(33a + 13b \pm \sqrt{a^2 - 78ab - 15b^2})$, w_2 and 0. This shows that $\mathbf{B}$ is diagonalizable except when $a/b = 39 \pm 16\sqrt{6}$. One can analyse these cases separately using a basis of generalized eigenvectors and show that the right hand side of Equation (6.14) still converges to $\mathbf{e}_{w_2}$.

3) Fix a maximal degree D. Choose partitioning weights

$$\begin{aligned} w_{1,i} &= w_{i,1} = (i-1)^{-1}, \quad i = 2, \ldots, D, \\ w_{2,D} &= w_{D,2} = D^{-1} \end{aligned}$$

and all other weights equal to zero. The splitting weights are then $w_i = 1$ for $i = 1, \ldots, D$. These weights satisfy the conditions in Lemma 6.1.3. Note that if we take the limit $D \longrightarrow \infty$ we get a special case of the preferential attachment model. The nonzero matrix elements of $\mathbf{B}$ are

$$B_{i+1,i} = B_{1,i} = -B_{i,i} = B_{2,1} = B_{2,D} = 1, \quad 1 < i < D. \tag{6.31}$$

The characteristic polynomial of $\mathbf{B}$ is

$$p_D(\lambda) = (-1)^D (1-\lambda) \left(1 - (1+\lambda)^{D-1}\right) \tag{6.32}$$

which can easily be proved by induction. The roots of the characteristic polynomial are $\lambda = 1$ and $\lambda = \exp\left(\frac{2\pi ik}{D-1}\right) - 1$, $k = 1, \ldots, D-1$ and they are all distinct which shows that $\mathbf{B}$ is diagonalizable. The solution to (6.27) is

$$\rho_k = \frac{2^{D-k+\delta_{kD}-1} - \delta_{k1}}{2^{D-1}-1}, \quad k = 1, \ldots, D. \tag{6.33}$$

6.3 Generality of results

It is not obvious how restrictive the condition that $\mathbf{B}$ must be diagonalizable is regarding the collection of weights one can consider. In the previous subsection we saw that for $D = 3$ and $D = 4$ the condition was not very restrictive. Also we saw that for every D there is at least one choice of weights which satisfies the conditions in Lemma 6.1.3 and yields a diagonalizable matrix $\mathbf{B}$. We will now show that this guarantees that almost all weights give a diagonalizable $\mathbf{B}$.

Fix a maximal degree D. Let $\mathcal{B}_D$ be the set of matrices $\mathbf{B}$ which correspond to partitioning weights that give linear splitting weights and satisfy the conditions in Lemma 6.1.3. It is clear that if $\mathbf{B}, \mathbf{B}' \in \mathcal{B}_D$ then $t\mathbf{B} + (1-t)\mathbf{B}' \in \mathcal{B}_D$ for all $t \in [0, 1]$ and so $\mathcal{B}_D$ is convex. Let

$$\mathcal{B}'_D = \{\mathbf{B} \in \mathcal{B}_D \mid \mathbf{B} \text{ is diagonalizable}\}.$$

From the previous subsection we know that $\mathcal{B}'_D \neq \emptyset$. Since $\mathcal{B}_D$ is convex and $\mathcal{B}'_D \neq \emptyset$ then by [47, Corollary 1], $\mathcal{B}'_D$ is dense in $\mathcal{B}_D$ in the standard topology.

We believe that it is possible to extend the result of convergence of the right hand side of (6.14) to all partitioning weights giving linear splitting weights, relaxing both the condition of diagonalizability of $\mathbf{B}$ and condition (2) in Lemma 6.1.3. We also believe, in view of simulations, that Equation (6.27) even describes correctly the vertex degree distribution for non–linear splitting weights and for the case $D = \infty$. We will look at this more closely in the next two subsections.

6.4 Mean field equation for general weights

To generalize Equation (6.27) beyond the case of linear splitting weights we notice that Lemmas 6.1.2 and 6.1.3 do not rely on the linearity of the weights except in the conclusion of Lemma 6.1.3 where we show that $w = w_2$. We therefore conjecture that in general the limiting vertex degree densities are the unique positive solution to

$$\rho_k = -\frac{w_k}{w}\rho_k + \sum_{i=k-1}^{D} i\frac{w_{k,i+2-k}}{w}\rho_i, \tag{6.34}$$

subject to the constraints

$$\rho_1 + \ldots + \rho_D = 1 \tag{6.35}$$

$$w_1\rho_1 + \ldots + w_D\rho_D = w. \tag{6.36}$$

Recall that w is the unique simple positive eigenvalue of $\mathbf{B}$ defined in (6.13) with the largest real part of all the eigenvalues and ρ_k, $k = 1, \ldots, D$ are the components of the associated eigenvector with the proper normalization.

The existence and uniqueness of a positive solution to (6.34) satisfying (6.35) and (6.36) follows from the Perron–Frobenius argument in the proof of Lemma 2.2. In order to distinguish (6.34) from (6.27) we refer to it as the *mean field equation for vertex degree densities*. One can also arrive directly at this equation by assuming that for large t an equilibrium with small enough fluctuations is established, and then performing the splitting procedure on this equilibrium.

The solution to the mean field equation for the $D = 3$ model and uniform partitioning weights is

$$\rho_3 = \frac{7\alpha - \sqrt{\alpha\,(\alpha + 24\,\beta + 24)}}{6(2\alpha - \beta - 1)} \tag{6.37}$$

where $\alpha = \dfrac{w_2}{w_1}$ and $\beta = \dfrac{w_3}{w_1}$. Note that from the constraints we have $\rho_1 = \rho_3$ and $\rho_2 = 1 - 2\rho_3$. This solution (and solutions in general) only depends on the ratio of the weights. In Figure 6.1 we compare the above solution to simulations.

6.5 The $D = \infty$ model with linear weights

In this subsection we drop the assumption that there is an upper bound on the vertex degrees but we still assume that all vertex degrees are finite. If we assume that Equation (6.27) holds for $D = \infty$, then it is possible to find an exact solution in the case of linear splitting weights, $w_i = ai + b$, and uniform partitioning weights.

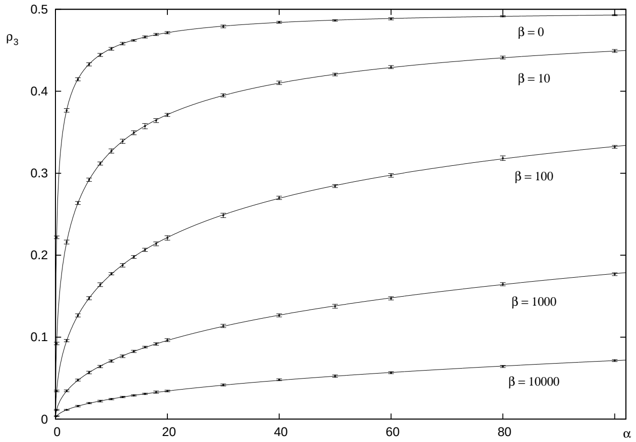


Figure 6.1: The value of ρ_3 as given in (6.37) compared to results from simulations. Each point is calculated from 20 trees with 10000 vertices.

Equation (6.27) becomes

$$\rho_k = -\frac{w_k}{w_2}\rho_k + \sum_{i=k-1}^{\infty} \frac{2}{i+1}\frac{w_i}{w_2}\rho_i. \tag{6.38}$$

Subtracting from this the same equation for ρ_{k+1} we find

$$\rho_k\left(1+\frac{w_k}{w_2}\right) - \rho_{k+1}\left(1+\frac{w_{k+1}}{w_2}\right) = \frac{2}{k}\frac{w_{k-1}}{w_2}\rho_{k-1}. \tag{6.39}$$

Let $x = b/a$. The recursion (6.39) has the solution

$$\rho_k(x) = \begin{cases} \dfrac{2}{C(-1)} & \text{if } x=-1 \text{ and } k=1 \\ \dfrac{1}{C(x)}\dfrac{2^{k-1}\Gamma(k+x)}{\Gamma(k)\Gamma(k+3+2x)}(k+1+2x) & \text{otherwise,} \end{cases} \tag{6.40}$$

where

$$C(x) = \frac{e\sqrt{\pi}\, 2^{-\frac{3}{2}-x} I_{\frac{1}{2}+x}(1)}{2+x} \tag{6.41}$$

is a normalization constant such that $\sum_i \rho_i = 1$. Here, I_ν is the modified Bessel function of the first kind. The variable x can take values from -1 to ∞. The asymptotic behaviour of $\rho_k(x)$ for large k is

$$\rho_k(x) = \frac{1}{C(x)} \frac{1}{k!} 2^{k-1} k^{-1-x} \left(1 + O\left(\frac{1}{k}\right)\right). \tag{6.42}$$

The special case $x = \infty$ corresponds to constant weights for which the solution is

$$\rho_k(\infty) = \frac{1}{e} \frac{1}{(k-1)!}. \tag{6.43}$$

In Figure 6.2 we compare the above solutions to simulations for five different values of x. The solid lines are $y = k + 1 + 2x$ plotted against k for five different values of x. The data points on the graph are calculated from simulations of 100 trees with 10^6 vertices. For a given k and x they are calculated from the degree densities of the simulated trees $\rho_{k,\mathrm{sim.}}(x)$ by

$$y = C(x) \frac{\Gamma(k)\,\Gamma(k+3+2x)}{2^{k-1}\Gamma(k+x)} \rho_{k,\mathrm{sim.}}(x) \tag{6.44}$$

with an obvious modification if $x = -1$.

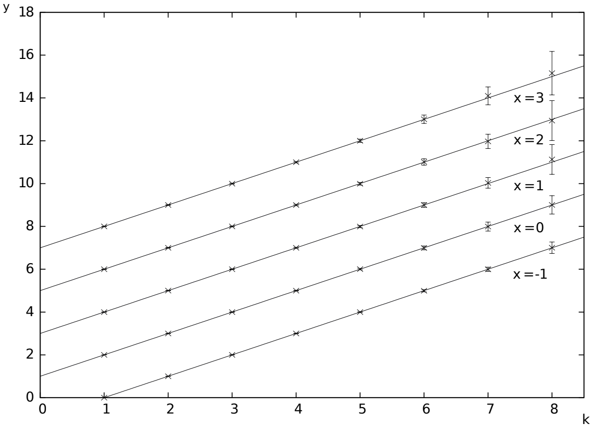


Figure 6.2: A comparison of Equation (6.40) to simulations.

7

Subtree structure probabilities and the Hausdorff dimension

In this chapter we consider the model in which vertices are labelled with their time of creation as explained in the definition of the splitting process (item (iv)). For convenience we will start from the single vertex tree at time 0. We consider only linear splitting weights $w_i = ai + b$ but comment on generalizations in the last section.

We derive exact expressions for probabilities of particular subtree structures as seen from the vertex created at a given time. By averaging over these probabilities and assuming the existence of a scaling limit, we shall show how to extract the Hausdorff dimension of the trees, as defined in (1.25), and derive bounds on this dimension. In special cases we give an exact expression for the Hausdorff dimension.

7.1 Volume distribution functions

Consider a tree of ℓ edges generated with the splitting procedure starting from the single vertex tree at time 0. To simplify the notation we define

$$W(\ell) \equiv \mathcal{W}(T) - w_1 = (2a+b)\ell - a \tag{7.1}$$

where the last equality follows from the linearity of the weights. This is the total weight of splitting a vertex in a tree T, excluding the root vertex (or any other leaf in fact). Let $p_R(\ell; s)$ be the probability that the vertex created at time s is the root.

If $s < \ell$ we find that

$$p_R(\ell; s) = \frac{1}{W(\ell-1)+w_1} W(\ell-1) p_R(\ell-1; s), \tag{7.2}$$

since we can split any vertex except the root in order to get from a tree at time $\ell - 1$ to a tree at time ℓ. This contributes the factor $p_R(\ell-1; s)$ to $p_R(\ell; s)$. Similarly,

$$p_R(\ell; \ell) = \frac{1}{W(\ell-1)+w_1} \sum_{s=0}^{\ell-1} w_1 p_R(\ell-1; s), \tag{7.3}$$

since if we create a new root vertex at time ℓ the previous root vertex, labelled s in (7.3) could have been created at any time before ℓ. We depict these processes in Fig. 7.1.

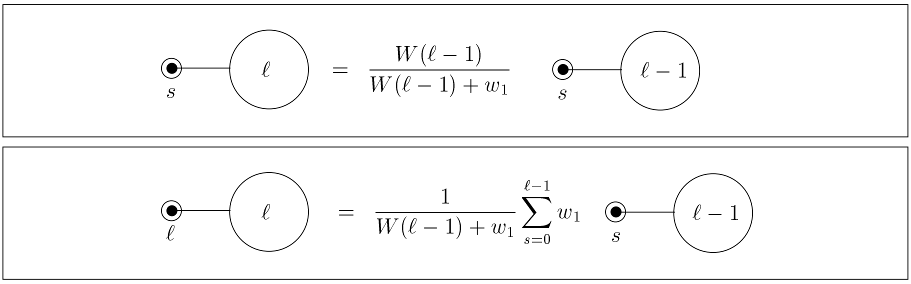


Figure 7.1: Diagrams representing equations (7.2) and (7.3).

If v is a vertex of order k in a tree T, then there is a unique edge e_1 incident on v leading towards the root (unless v is the root). Let $e_2, \ldots, e_k$ be the other edges incident on v. The largest subtree of T which contains the root and e_1 but none of the links e_i with $i \geq 2$ will be called the *left subtree* (with respect to v). The maximal subtrees which contain one e_j with $j \neq 1$ and no other link e_i will be called the *right subtrees* (with respect to v). If $k = 1$ then there are of course no right subtrees and if v is the root then we view the left subtree as being empty. Let $p_k(\ell_1, \ldots, \ell_k; s)$ denote the probability that the vertex created at time s has a left subtree of ℓ_1 edges and right subtrees of $\ell_2, \ldots, \ell_k$ edges, where $\ell_1 + \ldots + \ell_k = \ell$. By the nature of the splitting operation and because of the initial conditions, $p_k(\ell_1, \ell_2, \ldots, \ell_k; s)$ is symmetric under permutations of $(\ell_2, \ldots, \ell_k)$. We will sometimes refer to the vertex created at time s as the s-vertex.

By the definition of the relabelling when we split we have

$$p_1(\ell;\ell) = 0, \tag{7.4}$$

because the vertex closer to the root gets a new label and therefore no leaf except the root can have the maximal label. In the case $s < \ell$ we find the recursion

$$\begin{aligned} p_1(\ell;s) \quad &= \quad \frac{1}{W(\ell-1)+w_1}\Big[W(\ell-1)p_1(\ell-1;s) \\ &\quad + \sum_{i=1}^{D-1} i w_{i+1,1} \sum_{\ell'_1+\ldots+\ell'_i=\ell-1} p_i(\ell'_1,\ldots,\ell'_i;s) + \delta_{\ell 1} w_1\Big]. \end{aligned} \tag{7.5}$$

The first term in the square bracket corresponds to the case when we do not split the vertex with label s. The second term corresponds to splitting the s-vertex which can have any order up to $D-1$. Finally the last term corresponds to the special case when we have $\ell = 1$ so the s-vertex is the root of the trivial tree, see Fig. 7.2.

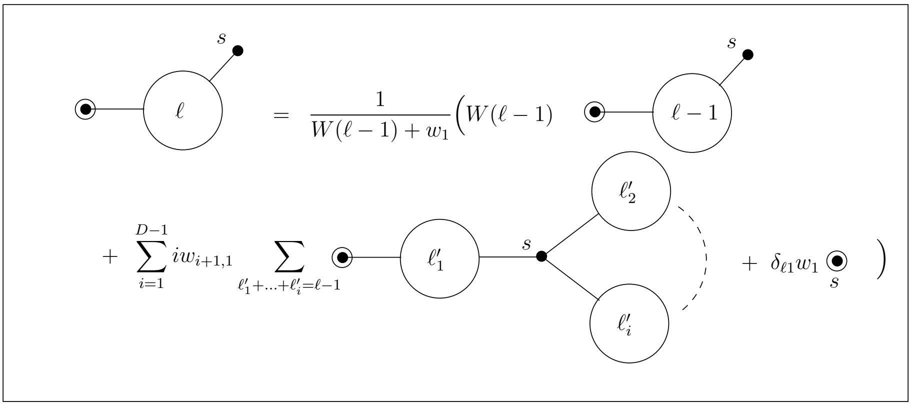


Figure 7.2: A diagram representing Equation (7.5).

For a general $k \geq 2$ and $s < \ell$ the recursion can be written

$$\begin{aligned} p_k(\ell_1,\ldots,\ell_k;s) \quad &= \quad \frac{1}{W(\ell-1)+w_1}\times \\ \Big[\delta_{k2}\delta_{\ell_1 1} w_1 p_R(\ell-1;s) &+ \sum_{i=1}^{k} W(\ell_i-1)p_k(\ell_1,\ldots,\ell_i-1,\ldots,\ell_k;s) \end{aligned}$$

$$+\sum_{i=k}^{D}(i+1-k)\,w_{k,i-k+2}\sum_{\ell_1'+\ldots+\ell_{i+1-k}'=\ell_1-1}p_i(\ell_1',\ldots,\ell_{i+1-k}',\ell_2,\ldots,\ell_k;s)\Big],\tag{7.6}$$

see Fig. 7.3. The first term corresponds to the case when the s-vertex is the root before the splitting in which case we have $\ell_1 = 1$ and $k = 2$. The second term corresponds to the case when we split a vertex different from the s-vertex and the last term arises when we split the s-vertex in the step from time $\ell - 1$ to time ℓ.

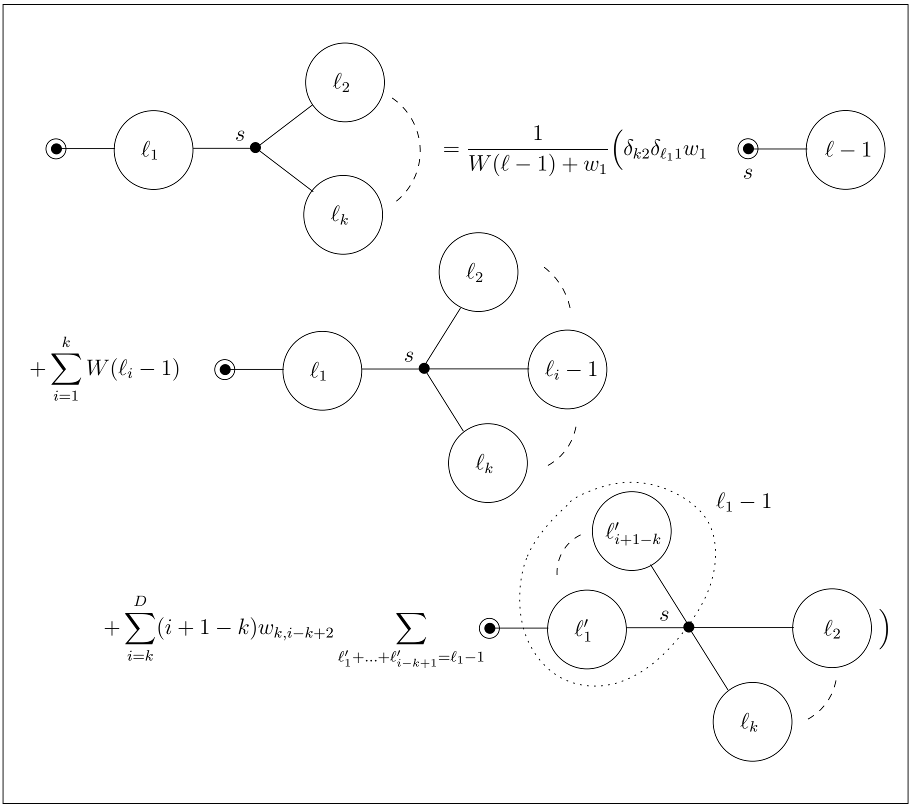


Figure 7.3: A diagram representing Equation (7.6).

Finally we have

$$p_k(\ell_1,\dots,\ell_k;\ell) = \frac{1}{W(\ell-1)+w_1}\times \sum_{s=0}^{\ell-1}\sum_{j=2}^{k}\sum_{i=k-1}^{D-1}\sum_{\substack{\ell'_1+\ldots+\ell'_{i+1-k}\\=\ell_j-1}} w_{k,i-k+2}\,p_i(\ell_1,\dots,\ell_{j-1},\ell'_1,\dots,\ell'_{i+1-k},\ell_{j+1},\dots,\ell_k;s), \tag{7.7}$$

where $\ell_1+\ldots+\ell_k=\ell$, see Fig.7.4. Here s is the label of the vertex that is split in the step from time $\ell-1$ to time ℓ and we sum over all possible degrees of the s-vertex and all ways of splitting it.

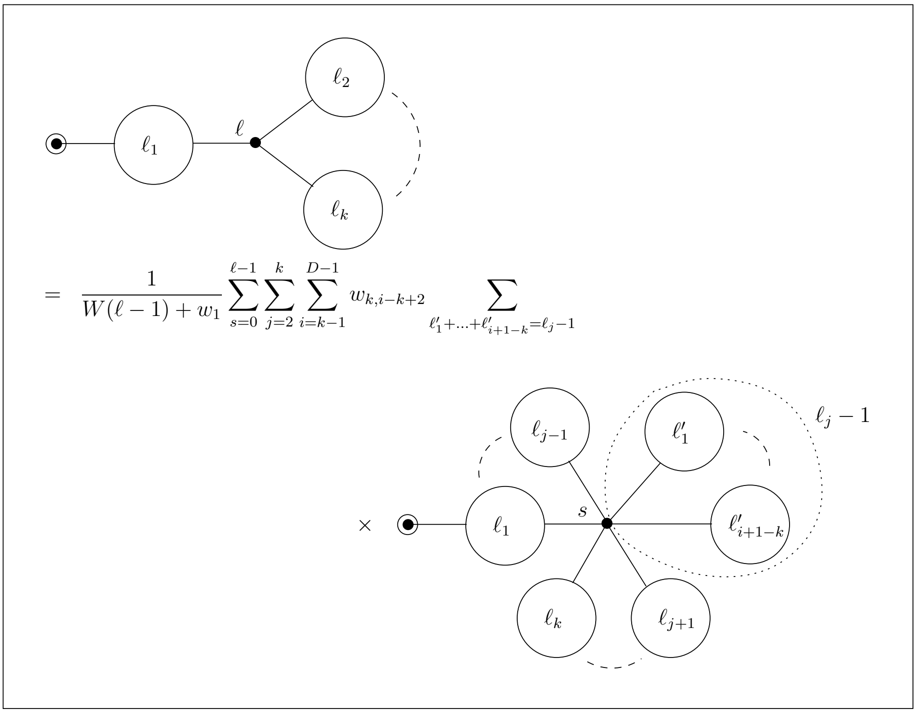


Figure 7.4: A diagram representing Equation (7.7).

We define the following mean probabilities by averaging over the vertex labels in (7.2–7.7)

$$p_R(\ell) = \frac{1}{\ell+1}\sum_{s=0}^{\ell} p_R(\ell;s) \tag{7.8}$$

and

$$p_k(\ell_1, \ldots, \ell_k) = \frac{1}{\ell+1} \sum_{s=0}^{\ell} p_k(\ell_1, \ldots, \ell_k; s), \tag{7.9}$$

where $\ell_1 + \ldots + \ell_k = \ell$. We refer to these functions as *volume distribution functions*. From (7.8) we get a recursion for the volume distribution functions, going from time ℓ to $\ell + 1$

$$p_R(\ell+1) \quad = \quad \frac{\ell+1}{\ell+2} p_R(\ell). \tag{7.10}$$

For $k = 1$ we obtain from (7.4), (7.5) and (7.9)

$$\begin{aligned} & p_1(\ell+1) \\ = \quad & \frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big[W(\ell)p_1(\ell) + \sum_{i=1}^{D-1} i w_{i+1,1} \sum_{\substack{\ell'_1+\ldots+\ell'_i \\ =\ell}} p_i(\ell'_1, ..., \ell'_i) + 2\delta_{\ell 0} w_1\Big]. \end{aligned} \tag{7.11}$$

Finally, the general case for $k \geq 2$ is

$$\begin{aligned} & p_k(\ell_1, \ldots, \ell_k) \\ = \quad & \frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big[\delta_{k2}\delta_{\ell_1 1} w_1 p_R(\ell) + \sum_{i=1}^{k} W(\ell_i - 1) p_k(\ell_1, \ldots, \ell_i - 1, \ldots, \ell_k) \\ & + \sum_{i=k}^{D} (i-k+1)\, w_{k,i-k+2} \sum_{\substack{\ell'_1+\ldots+\ell'_{i+1-k} \\ =\ell_1-1}} p_i(\ell'_1, \ldots, \ell'_{i+1-k}, \ell_2, \ldots, \ell_k) \\ & + \sum_{j=2}^{k} \sum_{i=k-1}^{D} w_{k,i-k+2} \sum_{\substack{\ell'_1+\ldots+\ell'_{i+1-k} \\ =\ell_j-1}} p_i(\ell_1, \ldots, \ell_{j-1}, \ell'_1, \ldots, \ell'_{i+1-k}, \ell_{j+1}, \ldots, \ell_k)\Big] \end{aligned} \tag{7.12}$$

where $\ell_1 + \ldots + \ell_k = \ell + 1$ and we have made use of (7.6), (7.7) and (7.9).

7.2 Geodesic distances and two point functions

One can reduce the above recursion formulas for the volume distribution functions to simpler recursion formulas which suffice for the determination of the Hausdorff dimension. Define the *two-point functions*

$$q_{ki}(\ell_1, \ell_2) = \sum_{\ell'_1+\ldots+\ell'_{k-i}=\ell_1} \; \sum_{\ell''_1+\ldots+\ell''_i=\ell_2} p_k(\ell'_1, \ldots, \ell'_{k-i}, \ell''_1, \ldots, \ell''_i), \tag{7.13}$$

where $k = 2, \ldots, D$ and $i = 1, \ldots, k-1$. In total there are $D(D-1)/2$ of these functions. If we define

$$q_{1,0}(\ell_1, \ell_2) = \delta_{\ell_2 0} \delta_{\ell_1 \ell} p_1(\ell_1 + \ell_2)$$

then $q_{ki}(\ell_1, \ell_2)$ is the probability that i right trees of total volume ℓ_2 are attached to a vertex of degree k in a tree of total volume $\ell_1 + \ell_2$. By summing over the equations in the previous section we get

$$\begin{aligned} q_{ki}(\ell_1, \ell_2) \quad = \quad & \frac{\ell+1}{\ell+2} \frac{1}{W(\ell)+w_1} \Big[\\ & \sum_{j=k-1}^{D} w_{k,j+2-k} \Big((j-i) q_{ji}(\ell_1 - 1, \ell_2) + i q_{j,j-(k-i)}(\ell_1, \ell_2 - 1) \Big) \\ & + \Big(W(\ell_1 - 1) + (k-i-1)(w_2 - w_3) \Big) q_{ki}(\ell_1 - 1, \ell_2) \\ & + \Big(W(\ell_2 - 1) + (i-1)(w_2 - w_3) \Big) q_{ki}(\ell_1, \ell_2 - 1) \\ & + \delta_{k2} \delta_{\ell_1 1} w_1 p_R(\ell_2) + \delta_{i1} \delta_{\ell_2 1} w_{k,1} \sum_{\ell_1' + \ldots + \ell_{k-1}' = \ell_1} p_{k-1}(\ell_1', \ldots, \ell_{k-1}') \Big] \end{aligned} \tag{7.14}$$

with $\ell_1 + \ell_2 = \ell + 1$. We see that the two-point functions satisfy an essentially closed system of equations. The last two terms in (7.14) do not contribute to the scaling limit which will be discussed in the next section.

The radius R_T defined in (1.24) can be extracted from these two point functions. Let T be a tree of ℓ edges and chose a $v \in V(T)$ and an $e \in E(T)$. If we cut the edge e at the vertex further away from v then the tree is split into two connected components, a tree T_1 which contains v and a tree T_2 that does not contain v (see Figure 7.5). Let $\ell_2(v; e)$ be the number of edges of T_2. We have the simple result

Lemma 7.2.1

$$\sum_{w \in V(T)} d_T(v, w) \sigma_T(w) \quad = \quad \sum_{e \in E(T)} (2\ell_2(v; e) + 1). \tag{7.15}$$

Proof For the tree T with ℓ edges, we may assign two labels to every edge in the following way. Starting from v, we walk around the tree while always keeping the tree to the left. Drop the labels 1 to 2ℓ on the sides of edges as we pass them.

An example of such a walk and labelling is shown in Figure 7.6. Let us mention

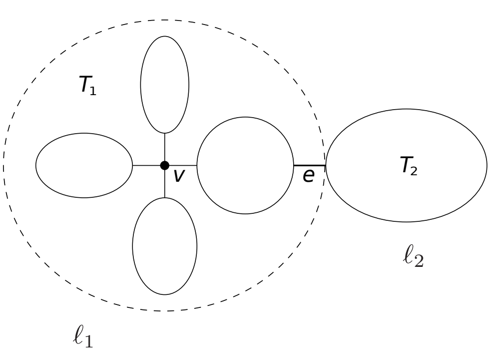


Figure 7.5: Cutting a tree along the edge e.

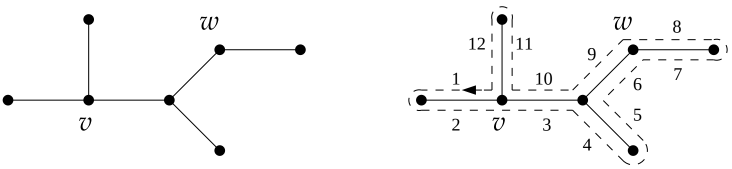


Figure 7.6: A tree and its labels.

that the initial direction from v is unimportant. In what follows we will denote these new labels by Greek letters.

Given $1 \leq \alpha < \beta \leq 2\ell$, define $\phi_v(\alpha,\beta)$ to be 1 if α and β are labels of the same edge, and zero otherwise. In the above example we have $\phi_v(6,9) = 1$ whereas $\phi_v(6,12) = 0$. For any vertex $w \in T$, let us define $\omega(w)$ to be the smallest label of the edges adjacent to w. In the example above $\omega(w) = 6$ and $\omega(v) = 1$. We now have for any $w \in T$

$$d_T(v,w) \quad = \sum_{\alpha,\beta\colon \alpha\leq\omega(w)<\beta} \phi_v(\alpha,\beta) \tag{7.16}$$

and it follows that

$$\sum_{w\in V(T)} d_T(v,w)\sigma_T(w) \quad = \sum_{\alpha,\beta,\gamma\colon \alpha\leq\gamma<\beta} \phi_v(\alpha,\beta) \;=\; \sum_{\alpha,\beta\colon \alpha<\beta} \phi_v(\alpha,\beta)(\beta-\alpha). \tag{7.17}$$

If $\phi(\alpha,\beta) = 1$, i.e. if α and β correspond to the two faces of the edge e, then

$$\beta - \alpha = 2\,\ell_2(v;e) + 1 \tag{7.18}$$

and Equation (7.15) follows.

□

We now apply (7.15) by choosing for v the root r of the tree and averaging over all trees obtained by the splitting process. We notice that the average number of links giving the volume $\ell_2(r;e)$ is simply the number of vertices, $\ell+1$, times the proportion of vertices which have a left tree (containing the root r) of $\ell_1 = \ell - \ell_2$ edges and an arbitrary number of right trees (with a total number of ℓ_2 edges). This proportion is precisely given by

$$\sum_{k=1}^{D} q_{k,k-1}(\ell - \ell_2, \ell_2). \tag{7.19}$$

Note that $\max\{\ell_2(r;e) \ : \ e \in T\} = \ell - 1$. We therefore obtain

$$\langle R_T \rangle_{\nu_\ell} = \frac{\ell+1}{2\ell} \sum_{\ell_2=0}^{\ell-1} (2\ell_2 + 1) \sum_{k=1}^{D} q_{k,k-1}(\ell - \ell_2, \ell_2). \tag{7.20}$$

Thus, if we know how the two point functions $q_{ki}(\ell_1,\ell_2)$ scale for large ℓ, we know how the radius of the tree scales with ℓ and we can compute the Hausdorff dimension d_H.

7.3 Scaling and the Hausdorff dimension

We assume that the following scaling holds for the two-point functions q_{ki} with ℓ large

$$q_{ki}(\ell_1,\ell_2) \quad = \quad \ell^{-\rho}\left(\overline{\omega}_{ki}(x) + \overline{\gamma}_{ki}(x)\ell^{-1} + O(\ell^{-2})\right) \tag{7.21}$$

where $\ell_1 + \ell_2 = \ell$, $x = \ell_1/\ell \in]0,1[$ and where $\overline{\omega}_{ki}, \overline{\gamma}_{ki}$ are some functions. It must hold that $\overline{\omega}_{ki} > 0$ and we assume that the scaling exponent ρ satisfies

$$1 < \rho \leq 2. \tag{7.22}$$

Note that for ℓ finite, the probabilities $q_{ki}(\ell_1,\ell_2)$ are of order ℓ^{-1} when ℓ_1 is of order 1 and are of order 1 when ℓ_2 is of order 1. This implies that the scaling functions

$\varpi_{ki}(x)$ should scale when $x \longrightarrow 0$ or $x \longrightarrow 1$, respectively, as

$$\varpi_{ki}(x) \sim x^{1-\rho} \quad \text{and} \qquad \varpi_{ki}(x) \sim (1-x)^{-\rho}. \tag{7.23}$$

Using this ansatz and (7.20) the mean radius scales as

$$\langle R_T(r)\rangle \simeq \ell^{2-\rho}\, C, \qquad C = \int_0^1 dx\,(1-x)\,\varpi(x), \qquad \varpi(x) = \sum_k \varpi_{k,k-1}(x). \tag{7.24}$$

Equations (7.23) and (7.22) ensure that the integral C is convergent when $\rho < 2$. Equation (1.25) then implies that the Hausdorff dimension of the tree is given by

$$2 - \rho = \frac{1}{d_H}. \tag{7.25}$$

For $\rho = 2$ we see that C is logarithmically divergent and this corresponds to an infinite Hausdorff dimension. Inserting (7.21) into the recursion Equation (7.14) for the two point functions and expanding in ℓ^{-1} gives

$$\begin{aligned}
&\varpi_{ki} - \rho\varpi_{ki}\ell^{-1} - x\ell^{-1}\varpi'_{ki} + \overline{\gamma}_{ki}\ell^{-1} + O(\ell^{-2}) \\
&= \frac{1}{w_2}\ell^{-1}\Big(1 - \frac{w_1 + 2w_2 - w_3}{w_2}\ell^{-1} + O(\ell^{-2})\Big) \\
&\quad \times\Big[\sum_{j=k-1}^{D} w_{k,j+2-k}\Big((j-i)\varpi_{ji} + i\varpi_{j,j-(k-i)} + O(\ell^{-1})\Big) \\
&\quad +\ell\Big(w_2 x + (-w_3 + (k-i-1)(w_2-w_3))\ell^{-1}\Big)\Big(\varpi_{ki} - \ell^{-1}\varpi'_{ki} + \overline{\gamma}_{ki}\ell^{-1} + O(\ell^{-2})\Big) \\
&\quad +\ell\Big(w_2(1-x) + i(w_2-w_3)\ell^{-1}\Big)\Big(\varpi_{ki} + \overline{\gamma}_{ki}\ell^{-1} + O(\ell^{-2})\Big)\Big].
\end{aligned} \tag{7.26}$$

where the $'$ denotes differentiation with respect to x and we have dropped the function argument x in an obvious way. The equation is trivially satisfied in zeroth order of ℓ^{-1}. When we go to the next order we see that the following must hold

$$(2-\rho)\varpi_{ki} \;=\; \frac{1}{w_2}\sum_{j=k-1}^{D} w_{k,j+2-k}\Big((j-i)\varpi_{ji} + i\varpi_{j,j-(k-i)}\Big) - \frac{w_k}{w_2}\varpi_{ki}. \tag{7.27}$$

This eigenvalue equation may be rewritten as

$$\mathbf{C}\boldsymbol{\omega} \;=\; w_2(2-\rho)\boldsymbol{\omega} \tag{7.28}$$

where $\mathbf{C}$ is a $\binom{D}{2} \times \binom{D}{2}$ matrix indexed by a pair of two indices ki with $k > i,\ k = 2, \ldots, D$ and $\boldsymbol{\omega}$ is a vector with two such indices. The matrix elements of $\mathbf{C}$ are

$$C_{ki,jn} \quad = \quad w_{k,j+2-k}\left((j-i)\,\delta_{in} + i\delta_{n,j-(k-i)}\right) - w_k\delta_{kj}\delta_{in}. \tag{7.29}$$

We use the convention that $w_{i,j} = 0$ if i or j is less than 1 or greater than D. Thus, $w_2(2-\rho)$ is an eigenvalue of the matrix $\mathbf{C}$ and the associated eigenvector must have components ≥ 0. We now show that there is in general a unique solution to this eigenvalue problem.

Since the only possibly negative elements of $\mathbf{C}$ are on the diagonal we can make the matrix nonnegative by adding a positive multiple γ of the identity to both sides of (7.28) and choosing γ large enough.

If enough of the weights $w_{i,j}$ are nonzero ($w_{1,i} > 0$ for $2 \leq i \leq D$ and $w_{j,3} > 0$ for $2 \leq j \leq D-1$ is for example sufficient) then one can check that the matrix $\mathbf{C} + \gamma\mathbf{I}$ is primitive. Then, by the Perron–Frobenius theorem, it has a simple positive eigenvalue of largest modulus and its corresponding eigenvector can be taken to have all entries positive cf. Lemma 6.1.3. Therefore this largest positive eigenvalue gives the ρ we are after.

7.3.1 An upper bound on the Hausdorff dimension

We can get an upper bound on ρ by a straightforward estimate from (7.27). The off-diagonal terms in the sum are all nonnegative so we disregard them and get the inequality

$$\rho \quad \leq \quad 2 - \left(k\frac{w_{k,2}}{w_2} - \frac{w_k}{w_2}\right), \qquad k = 2, \ldots, D. \tag{7.30}$$

Since $1 < \rho \leq 2$ and for $k \geq 3$

$$w_k \quad = \quad kw_{k,2} + \frac{k}{2}\sum_{\substack{i=1 \\ i\neq 2,\ i\neq k}}^{k+1} w_{i,k+2-i} \ > \ kw_{k,2} \tag{7.31}$$

the best we can get from this upper bound is when $k = 2$ which yields

$$\rho \quad \leq \quad 2 - \frac{w_{2,2} - 2w_{1,3}}{w_{2,2} + 2w_{1,3}}. \tag{7.32}$$

Now, $2 - \rho = \frac{1}{d_H}$ and therefore, if $w_{2,2} > 2w_{1,3}$, we obtain the upper bound

$$d_H \quad \leq \quad \frac{w_{2,2} + 2w_{1,3}}{w_{2,2} - 2w_{1,3}}. \tag{7.33}$$

If $w_{2,2} \leq 2w_{1,3}$ the upper bound in (7.32) gives no information about the Hausdorff dimension. The condition $w_{2,2} > 2w_{1,3}$ means that splittings of vertices of degree 2 which lengthen the tree are more frequent than the splittings of vertices of degree 2 which increase the branching of the tree. It is interesting to note that this condition between "stretching" and branching of vertices of degree 2 is enough to provide a finite Hausdorff dimension.

It is easy to verify that (7.33) is an equality if we choose the weights such that $w_{i,j} = 0$ if $i \neq 1$ or $j \neq 1$ with the exception that $w_{2,2} > 0$. This condition means that we only allow vertices to evolve by link attachment, except that we can split vertices of degree 2. With this choice the matrix $\mathbf{C}$ is lower triangular and we can simply read the eigenvalues from the diagonal. Note that $\mathbf{C}$ is not primitive in this case and therefore we cannot use the Perron–Frobenius theorem to determine which eigenvalue gives the scaling exponent. However, with these simple weights one can show explicitly that there is precisely one eigenvector with strictly positive components and the corresponding eigenvalue is the one that saturates the inequality in (7.33). Also note that with this choice we have set $w_D = 0$ and since the weights are linear, $w_i = ai + b$, we have fixed a and b so that $w_i = 1 - \frac{i}{D}$. Therefore there is only one free parameter which we can choose to be $w_{2,2}$. Then we can write the Hausdorff dimension as

$$d_H = \frac{1 - 2/D}{2w_{2,2} - (1 - 2/D)} \tag{7.34}$$

with

$$\frac{1}{2}(1 - 2/D) < w_{2,2} < 1 - 2/D. \tag{7.35}$$

We see that for any D the Hausdorff dimension can vary continuously from 1 to infinity.

7.3.2 Explicit solutions and numerical results for $D = 3$

When the maximal degree is $D = 3$, the splitting weights are taken to be linear $w_i = ai + b$ and the partitioning weights uniform, it is easy to solve Equation (7.27) for the Hausdorff dimension . Since the solution only depends on the ratio of the weights there is only one independent variable and we choose it to be $y := w_3/w_2$ where $0 \leq y \leq 2$. The solution is

$$d_H = \frac{3(1 + \sqrt{1 + 16y})}{8y} \tag{7.36}$$

In Figure 7.7 we compare this equation to results from simulations. The agreement of the simulations with the formula is good in the tested range $0.5 \leq y \leq 2$. For smaller values of y the Hausdorff dimension increases fast and one would have to simulate very large trees to see the scaling.

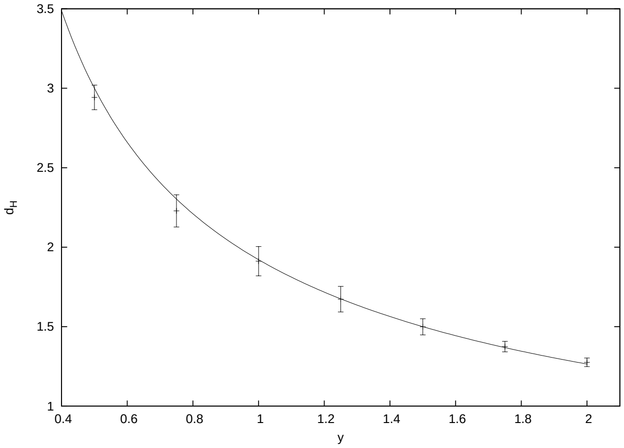


Figure 7.7: Equation (7.36) compared to simulations. The Hausdorff dimension, d_H, is plotted against $y = w_3/w_2$. The leftmost data point is calculated from 50 trees with 50000 vertices and the others are calculated from 50 trees with 10000 vertices.

7.3.3 General mean field argument

Our argument to compute the Hausdorff dimension relies on the recursion relations for the substructure probabilities, studied in Section 7.2, which are valid only when the splitting weights w_i are linear functions of the vertex degree i ($w_i = ai + b$). In this case the total probability weight $\mathcal{W}(T)$ for a given tree T depends only on its size ℓ (number of edges) and mean field arguments can be made exact.

In the general case where the $w_{i,j}$ are arbitrary and the w_i are not linear with i, these recursion relations are no longer exact. We can use a mean field argument and assume that they are still valid for large "typical" trees, provided that we replace in these recursion relations the exact weights $W(\ell) + w_1 = \mathcal{W}(T)$ by their mean field value for large trees

$$W(\ell) + w_1 \ \longrightarrow\ \overline{\mathcal{W}}(\ell) = \sum_j w_j \overline{n}_{\ell+1,j} \tag{7.37}$$

where $\overline{n}_{\ell+1,j}$ is the average number of i-vertices in a tree with ℓ edges, studied in Section 6.1. From the mean field analysis of Section 6.4, we expect that these $\overline{n}_{\ell+1,j}$ scale with ℓ as

$$\overline{n}_{\ell+1,j} \simeq \ell \rho_j \tag{7.38}$$

with the vertex densities ρ_j given by the mean field equations (6.34, 6.35, 6.36) as the components of the eigenvector $\boldsymbol{\rho}$ associated to the largest eigenvalue w of the matrix $\mathbf{B}$. Thus the mean field approximation amounts to replacing

$$W(\ell) + w_1 \ \longrightarrow\ \overline{\mathcal{W}}(\ell) = w\,\ell + \cdots \tag{7.39}$$

in the recursion relations of the previous sections, in particular in the recursion relation (7.14) for the two point function q_{ki}.

With this assumption, we can repeat the scaling argument of Section 7.3, and we end up with Equation (7.27), with the normalisation factor $\frac{1}{w_2}$ in the r.h.s. replaced by the mean field normalisation factor $\frac{1}{w}$

$$(2-\rho)\,\overline{\omega}_{ki} = \frac{1}{w} \sum_{j=k-1}^{D} w_{k,j+2-k} \Big((j-i)\overline{\omega}_{ji} + i\overline{\omega}_{j,j-(k-i)} \Big) - \frac{w_k}{w} \overline{\omega}_{ki} \ . \tag{7.40}$$

This equation is still an eigenvalue equation of the form

$$\mathbf{C}\,\boldsymbol{\omega} = w(2-\rho)\,\boldsymbol{\omega} \tag{7.41}$$

where $\mathbf{C}$ is the $\binom{D}{2} \times \binom{D}{2}$ matrix with coefficients given in (7.29).

If we denote by χ the largest eigenvalue of this matrix $\mathbf{C}$ and if w is the largest eigenvalue of $\mathbf{B}$ then the Perron–Frobenius argument can be applied to show that χ is nonnegative and that the eigenvector $\boldsymbol{\omega}$ has nonnegative components, which is a consistency requirement for the argument, since the ω_{ki} are rescaled probabilities. We end up with a mean field prediction for the Hausdorff dimension of the simple form

$$d_H = \frac{1}{2-\rho} = \frac{w}{\chi}. \tag{7.42}$$

General solution for $D = 3$

In the $D = 3$ case, the $\mathbf{B}$ and $\mathbf{C}$ matrices are

$$\mathbf{B} = \begin{bmatrix} 0 & 2w_{3,1} & 0 \\ w_{2,1} & w_{2,2} - 2\,w_{3,1} & 3\,w_{3,2} \\ 0 & 2\,w_{3,1} & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} w_{2,2} - 2\,w_{3,1} & 2w_{3,2} & w_{3,2} \\ w_{3,1} & 0 & 0 \\ 2\,w_{3,1} & 0 & 0 \end{bmatrix} \tag{7.43}$$

and we find

$$d_H = \frac{(w_{2,2} - 2w_{3,1}) + \sqrt{(w_{2,2} - 2w_{3,1})^2 + 8w_{3,1}(w_{2,1} + 3w_{3,2})}}{(w_{2,2} - 2w_{3,1}) + \sqrt{(w_{2,2} - 2w_{3,1})^2 + 16w_{3,1}w_{3,2}}}. \tag{7.44}$$

We have tested this formula when the partitioning weights $w_{i,j}$ are uniform. In this

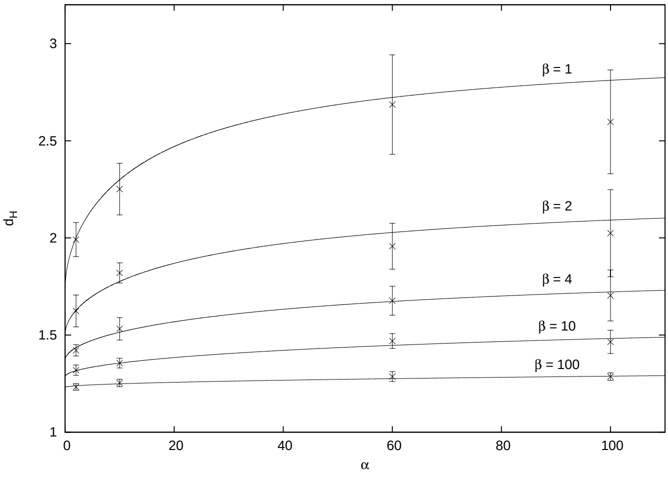


Figure 7.8: Equation (7.45) compared to simulations. Each data point is calculated from 50 trees with 10000 vertices.

case

$$d_H = \frac{\alpha - \sqrt{\alpha\,(\alpha + 24 + 24\,\beta)}}{\alpha - \sqrt{\alpha\,(\alpha + 16\,\beta)}} \tag{7.45}$$

where $\alpha = \frac{w_2}{w_1}$ and $\beta = \frac{w_3}{w_1}$. In Figure 7.8 we compare this equation to results from simulations. There is a good agreement for small values of α and β, but the precision of the numerics becomes poor for large values of α and β. This is expected since in this case, the trees will have a large Hausdorff dimension and one must go to very large trees to see the scaling.

8

Correlation between degrees of neigbouring vertices

Consider a tree of ℓ edges generated by the splitting procedure starting from the single vertex tree at time 0. We are interested in determining the density of edges which have endpoints of degrees j and k in the limit when $\ell \longrightarrow \infty$. We define this density in the following way. Distribute arrows uniformly at random to each edge of a tree and let $\bar{\rho}_{jk}$ denote the average density of edges which have an arrow pointing from a vertex of degree j to a vertex of degree k. A knowledge of the $\bar{\rho}_{jk}$ allows us to determine whether vertices of high degree prefer to be neighbours of vertices of high degree, in which case the tree is said to show *assortative mixing*, or whether they prefer to be neighbours of vertices of low degree, in which case the tree is said to show *disassortative mixing*. For instance, social networks often show assortative mixing whereas biological and technical networks tend to be disassortative [59].

First note that the degree distribution of an endpoint of a randomly chosen edge in a graph is proportional to $k\rho_k$ rather than ρ_k. We therefore define the densities

$$\bar{\rho}_k = \frac{k\rho_k}{\sum_i i\rho_i} \tag{8.1}$$

for general graphs, and in the case of trees $\sum_i i\rho_i = 2$. The amount of assortative mixing in a general graph is quantified by a correlation coefficient r which compares the densities $\bar{\rho}_{jk}$ to densities in graphs where no correlations are present, i.e. when

$\bar{\rho}_{jk} = \bar{\rho}_j \bar{\rho}_k$. The correlation coefficient is defined as

$$r = \frac{\sum_{j,k=1}^{D} (j-1)(k-1)(\bar{\rho}_{jk} - \bar{\rho}_j \bar{\rho}_k)}{\sum_{j=1}^{D} (k-1)^2 \bar{\rho}_k - \left(\sum_{k=1}^{D} (k-1) \bar{\rho}_k \right)^2} \tag{8.2}$$

where the denominator is chosen such that $r \in [-1, 1]$ [59]. When r is negative the graphs are disassortative and when r is positive they are assortative. Note that $r = 0$ for graphs for which $\bar{\rho}_{jk} = \bar{\rho}_j \bar{\rho}_k$ and such graphs are in general not connected since $\bar{\rho}_{1,1} \neq 0$ if $\bar{\rho}_1 \neq 0$. However, this does not hold in the other direction; for instance, the preferential attachment model with linear attachment weights and $D = \infty$ has $r = 0$ [59].

For convenience we first calculate the density of edges which have endpoints of degrees j and k such that the vertex of degree j is the one closer to the root. This density will be denoted by ρ_{jk}. It holds that $\rho_{1k} = 0$ for all k, in general $\rho_{jk} \neq \rho_{kj}$ and

$$\bar{\rho}_{jk} = \frac{\rho_{jk} + \rho_{kj}}{2}. \tag{8.3}$$

In the following sections we calculate ρ_{jk} using a scaling argument and compare the results in the case $D = 3$ to results from simulations. We conclude the chapter by discussing the amount of assortative mixing in the vertex splitting model. The model is disassortative for the range of parameters we consider except in the special case of the preferential attachment model with linear splitting weights for which $r = 0$.

8.1 Calculation of ρ_{jk}

To arrive at the densities ρ_{jk}, we use the same labelling techniques as in Chapter 7. To begin with, let us assume that the splitting weights are linear. Define

$$p_{jk}(\ell'_1, \ldots, \ell'_{j-1}; \ell''_1, \ldots, \ell''_{k-1}; s)$$

as the probability that a vertex created at time s is of degree k and has $\ell''_1, \ldots, \ell''_{k-1}$ right trees and that the vertex to its left is of degree j with an ℓ'_1 left tree and $\ell'_2, \ldots, \ell'_{j-1}$ right trees (excluding the right tree which contains s). Note that it is symmetric under permutations of both $(\ell'_2, \ldots, \ell'_{j-1})$ and $(\ell''_1, \ldots, \ell''_{k-1})$ and

$$\ell'_1 + \ldots + \ell'_{j-1} + \ell''_1 + \ldots + \ell''_{k-1} \quad = \quad \ell - 1.$$

We derive recursion equations for $p_{jk}(\ell'_1,\dots,\ell'_{j-1};\ell''_1,\dots,\ell''_{k-1};s)$ and all nontrivial equations are explained in Figures 8.1–8.5. To make the notation more compact we will write for $i \le j$

$$\ell_{i,j} = \ell_i,\dots,\ell_j, \quad \text{and} \quad |\ell_{i,j}| = \ell_i + \dots + \ell_j.$$

We can write the following recursions for going from time $\ell-1$ to time ℓ. Note that $s<\ell$ in (8.4), (8.6) and (8.8).

$$\begin{aligned} p_{1k}(\ell''_{1,k-1};s) \;=&\\ &\frac{1}{W(\ell-1)+w_1}\Big(\sum_{i=1}^{k-1} W(\ell''_i-1)p_{1k}(\ell''_{1,i-1},\ell''_i-1,\ell''_{i+1,k-1};s)\\ &\qquad\qquad + \;\delta_{k2}w_1p_R(\ell-1;s)+\delta_{\ell 1}\delta_{k1}w_1\Big). \end{aligned} \tag{8.4}$$

$$\begin{aligned} p_{1k}(\ell''_{1,k-1};\ell) \;=&\\ &\frac{1}{W(\ell-1)+w_1}\sum_{s=0}^{\ell-1}\sum_{i=k-1}^{D}(k-1)w_{k,i+2-k}\sum_{|\tilde\ell_{1,i+1-k}|=\ell''_1-1} p_{1i}(\tilde\ell_{1,i+1-k},\ell''_{1,k-1};s). \end{aligned} \tag{8.5}$$

$$\begin{aligned} p_{j1}(\ell'_{1,j-1};s) \;=&\\ &\frac{1}{W(\ell-1)+w_1}\Big(\sum_{i=1}^{j-1} W(\ell'_i-1)p_{j1}(\ell'_{1,i-1},\ell'_i-1,\ell'_{i+1,j-1};s)\\ &\qquad + \;(j-1)w_{j,1}p_{j-1}(\ell'_{1,j-1};s)\\ &+\sum_{i=j-1}^{D}\frac{2w_{j,i+2-j}}{i-1}\sum_{p=1}^{j-2}\sum_{n=p+1}^{j-1}\sum_{|\tilde\ell_{1,i+1-j}|=\ell'_n-1} p_{i1}(\ell'_{1,n-1},\tilde\ell_{1,i+1-j},\ell'_{n+1,j-1};s),\\ &+\;(j-1)\sum_{i=j}^{D}\frac{(i-j+1)w_{j,i+2-j}}{i-1}\sum_{|\tilde\ell_{1,i+1-j}|=\ell'_1-1} p_{i1}(\tilde\ell_{1,i+1-j},\ell'_{2,j-1};s)\Big), \end{aligned} \tag{8.6}$$

$$p_{j1}(\ell'_{1,j-1};\ell) \;=\; 0 \tag{8.7}$$

$$\begin{aligned} p_{jk}(\ell'_{1,j-1};\ell''_{1,k-1};s) \;=&\\ &\frac{1}{W(\ell-1)+w_1}\Big(\sum_{i=1}^{j-1} W(\ell'_i-1)p_{jk}(\ell'_{1,i-1},\ell'_i-1,\ell'_{i+1,j-1};\ell''_{1,k-1};s) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{k-1} W(\ell''_i - 1)\, p_{jk}(\ell'_{1,j-1}; \ell''_{1,i-1}, \ell''_i - 1, \ell''_{i+1,k-1}; s) \\
&+ (j-1) w_{j,k}\, p_{j+k-2}(\ell'_{1,j-1}, \ell''_{1,k-1}; s) \\
&+ \sum_{i=j-1}^{D} \frac{2w_{j,i+2-j}}{i-1} \sum_{p=1}^{j-2} \sum_{n=p+1}^{j-1} \sum_{|\tilde{\ell}_{1,i+1-j}|=\ell'_n-1} p_{ik}(\ell'_{1,n-1}, \tilde{\ell}_{1,i+1-j}, \ell'_{n+1,j-1}; \ell''_{1,k-1}; s) \\
&+ (j-1) \sum_{i=j}^{D} \frac{i-j+1}{i-1} w_{j,i+2-j} \sum_{|\tilde{\ell}_{1,i+1-j}|=\ell'_1-1} p_{ik}(\tilde{\ell}_{1,i+1-j}, \ell'_{2,j-1}; \ell''_{1,k-1}; s)\Big),
\end{aligned}
\tag{8.8}
$$

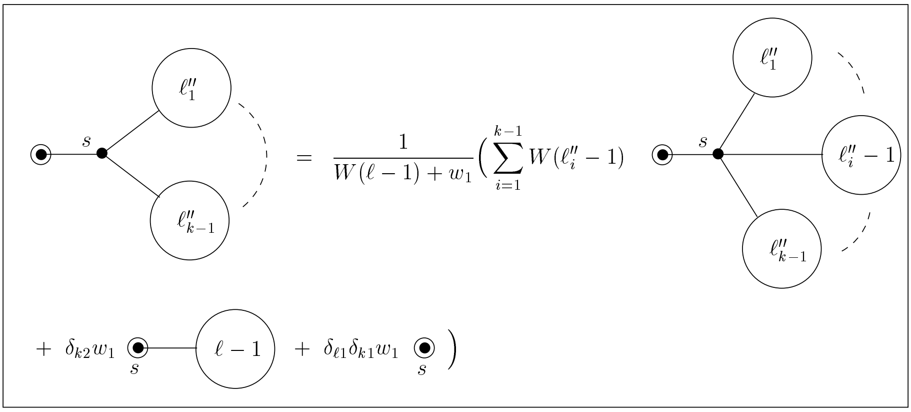


Figure 8.1: Illustration of Equation (8.4).

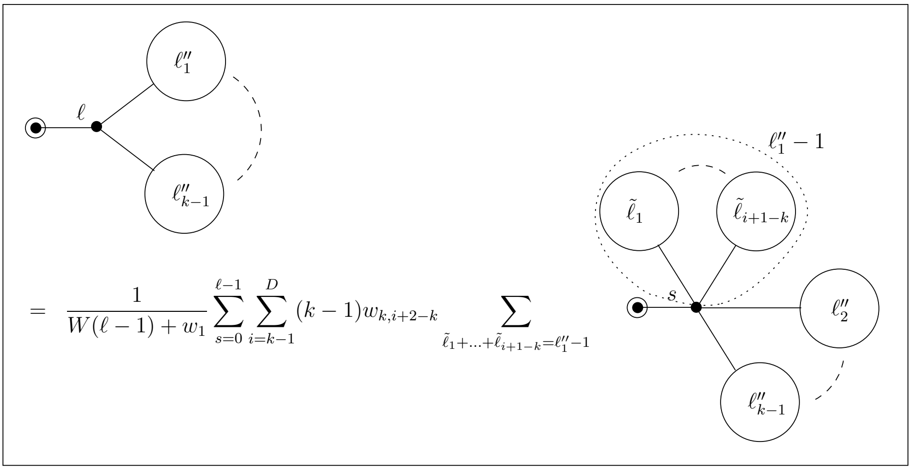


Figure 8.2: Illustration of Equation (8.5).

$$
\begin{aligned}
p_{jk}(\ell'_{1,j-1};\ell''_{1,k-1};\ell) &= \frac{1}{W(\ell-1)+w_1}\sum_{s=0}^{\ell-1}\sum_{n=1}^{k-1}\sum_{i=k-1}^{D} w_{k,i+2-k} \\
&\times \sum_{|\tilde{\ell}_{1,i+1-k}|=\ell''_n-1} p_{ji}(\ell'_{1,j-1};\ell''_{1,n-1},\tilde{\ell}_{1,i+1-k},\ell''_{n+1,k-1};s). \qquad (8.9)
\end{aligned}
$$

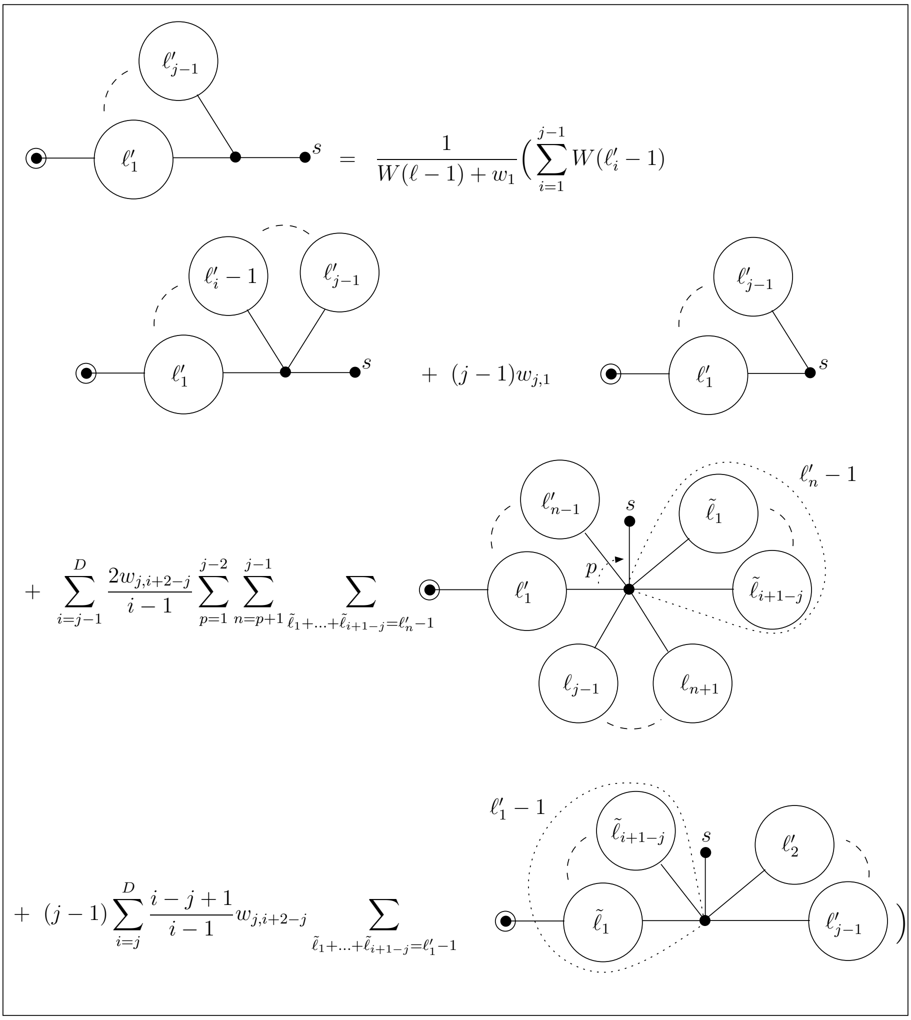


Figure 8.3: Illustration of Equation (8.6).

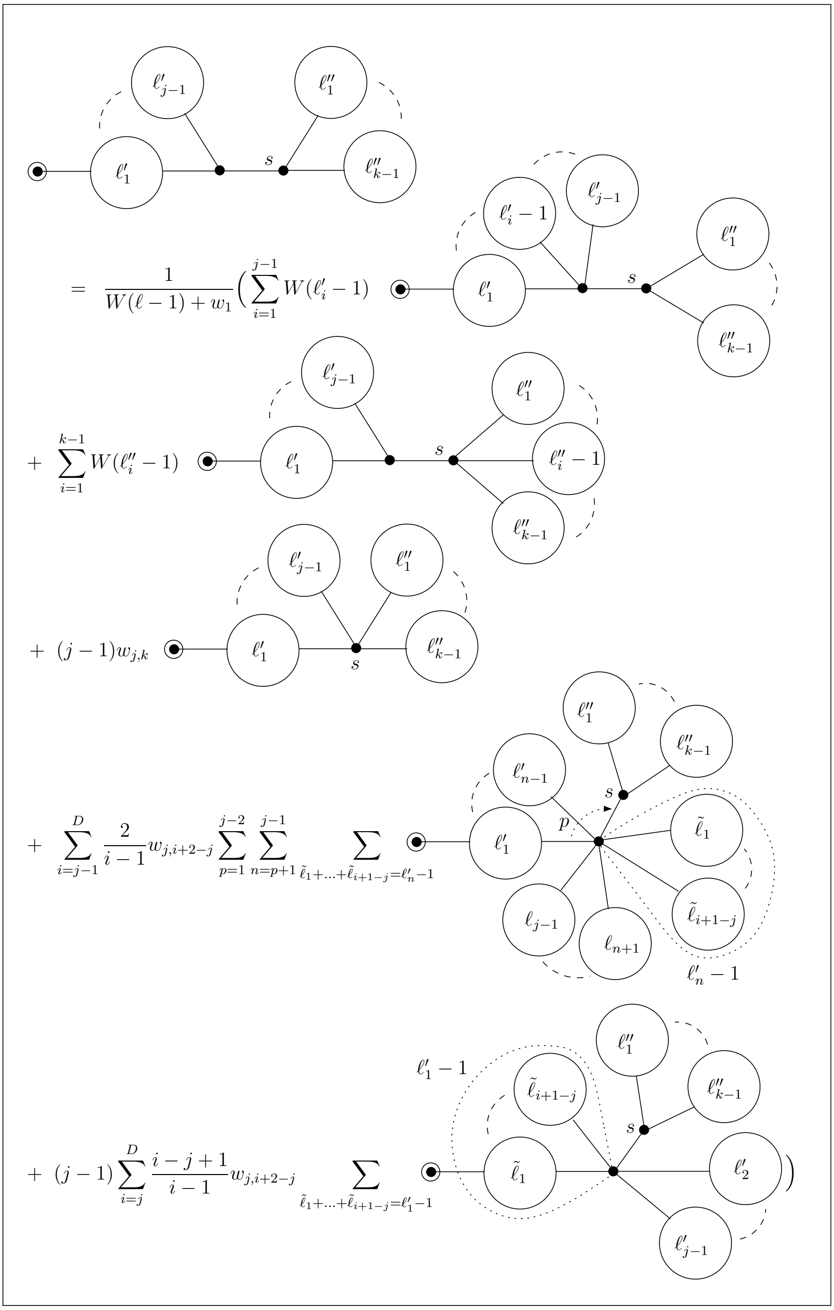


Figure 8.4: Illustration of Equation (8.8).

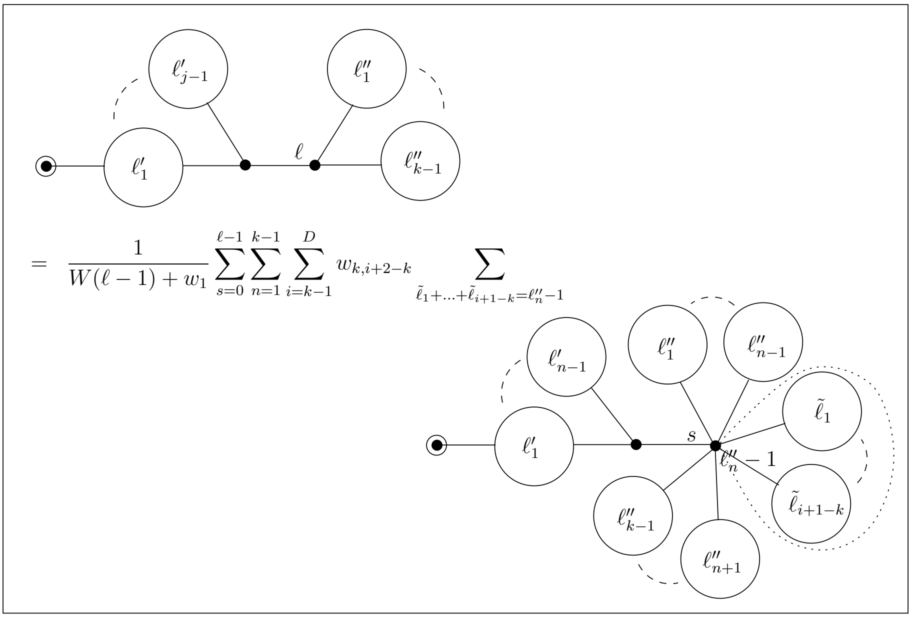


Figure 8.5: Illustration of Equation (8.9).

In deriving Equations (8.6) and (8.8) and the corresponding figures, note that the index p is introduced in the second last diagram in each figure. The reason for this is the following: even though $p_{j1}(\ell'_1, \ldots, \ell'_{j-1}; s)$ and $p_{jk}(\ell'_1, \ldots, \ell'_{j-1}; \ell''_1, \ldots, \ell''_{k-1}; s)$ are symmetric under permutations of $(\ell'_2, \ldots, \ell'_{j-1})$ it does matter where the edge going from s towards the root, is located. Therefore, we group together the balloons counter-clockwise from s towards the rooted balloon and we group together the balloons clockwise from s towards the rooted balloon, one of the groups is possibly empty. If the total number of balloons in the groups is $i-2$ then there are $i-1$ such possible configurations. In the equations we therefore divide by $i-1$ and sum over all the configurations which contribute to the configuration on the left of the equality sign. The index p in the sum is the location of s clockwise from the rooted balloon. Note that p can be no larger than $j-2$ since if it were larger, there would be no space for the balloons inside the dotted circle. Note that the balloons inside the dotted circle are always drawn clockwise from the vertex s. To count the possibility that they are counter-clockwise from s we multiply by 2.

Now average over the label s as before and get the following recursion, going from time ℓ to $\ell+1$

$$
\begin{aligned}
p_{1k}(\ell''_{1,k-1}) \;&= \\
&\frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big(\sum_{i=1}^{k-1} W(\ell''_i-1)p_{1k}(\ell''_{1,i-1},\ell''_i-1,\ell''_{i+1,k-1})+\delta_{k2}w_1p_R(\ell) \\
&+\;\delta_{\ell,0}\delta_{k1}w_1+\;(k-1)\sum_{i=k-1}^{D} w_{k,i+2-k}\sum_{|\tilde{\ell}_{1,i+1-k}|=\ell''_1-1} p_{1i}(\tilde{\ell}_{1,i+1-k},\ell''_{1,k-1})\Big).
\end{aligned}
$$

$$
\begin{aligned}
p_{j1}(\ell'_{1,j-1}) \;&= \\
&\frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big(\sum_{i=1}^{j-1} W(\ell'_i-1)p_{j1}(\ell'_{1,i-1},\ell'_i-1,\ell'_{i+1,j-1}) \\
&\qquad\qquad+\;(j-1)w_{j,1}p_{j-1}(\ell'_{1,j-1}) \\
&+\sum_{i=j-1}^{D}\frac{2w_{j,i+2-j}}{i-1}\sum_{p=1}^{j-2}\sum_{n=p+1}^{j-1}\sum_{|\tilde{\ell}_{1,i+1-j}|=\ell'_n-1} p_{i1}(\ell'_{1,n-1},\tilde{\ell}_{1,i+1-j},\ell'_{n+1,j-1}) \\
&+\;(j-1)\sum_{i=j}^{D}\frac{i-j+1}{i-1}w_{j,i+2-j}\sum_{|\tilde{\ell}_{1,i+1-j}|=\ell'_1-1} p_{i1}(\tilde{\ell}_{1,i+1-j},\ell'_{2,j-1})\Big).
\end{aligned}
$$

$$
\begin{aligned}
p_{jk}(\ell'_{1,j-1};\ell''_{1,k-1}) \;&= \\
&\frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big(\sum_{i=1}^{j-1} W(\ell'_i-1)p_{jk}(\ell'_{1,i-1},\ell'_i-1,\ell'_{i+1,j-1};\ell''_{1,k-1}) \\
&+\;\sum_{i=1}^{k-1} W(\ell''_i-1)p_{jk}(\ell'_{1,j-1};\ell''_{1,i-1},\ell''_i-1,\ell''_{i+1,k-1}) \\
&+\;(j-1)w_{j,k}p_{j+k-2}(\ell'_{1,j-1},\ell''_{1,k-1}) \\
&+\sum_{i=j-1}^{D}\frac{2w_{j,i+2-j}}{i-1}\sum_{p=1}^{j-2}\sum_{n=p+1}^{j-1}\sum_{|\tilde{\ell}_{1,i+1-j}|=\ell'_n-1} p_{ik}(\ell'_{1,n-1},\tilde{\ell}_{1,i+1-j},\ell'_{n+1,j-1};\ell''_{1,k-1}) \\
&+\;(j-1)\sum_{i=j}^{D}\frac{i-j+1}{i-1}w_{j,i+2-j}\sum_{|\tilde{\ell}_{1,i+1-j}|=\ell'_1-1} p_{ik}(\tilde{\ell}_{1,i+1-j},\ell'_{2,j-1};\ell''_{1,k-1}) \\
&+\;\sum_{n=1}^{k-1}\sum_{i=k-1}^{D} w_{k,i+2-k}\sum_{|\tilde{\ell}_{1,i+1-k}|=\ell''_n-1} p_{ji}(\ell'_{1,j-1};\ell''_{1,n-1},\tilde{\ell}_{1,i+1-k},\ell''_{n+1,k-1})\Big).
\end{aligned}
$$

Finally we define the densities $\rho_{jk}(\ell)$ by averaging out the volume dependence of the average probabilities

$$\rho_{jk}(\ell) \quad = \sum_{\ell'_1+\ldots\ell'_{j-1}+\ell''_1+\ldots+\ell''_{k-1}=\ell-1} p_{jk}(\ell'_1,\ldots,\ell'_{j-1};\ell''_1,\ldots,\ell''_{k-1})$$

and similarly we denote the vertex degree density by

$$\rho_j(\ell)\equiv\rho_{\ell,j} \quad = \sum_{\ell_1+\ldots+\ell_j=\ell} p_j(\ell_1,\ldots,\ell_j),$$

cf. Section 6.1. We have the following recursion for the densities

$$\begin{aligned}\rho_{jk}(\ell+1) \quad &= \\ &\frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Bigg\{(\ell w_2-w_j-w_k+2w_1-w_2)\rho_{jk}(\ell)+(j-1)w_{j,k}\rho_{j+k-2}(\ell)\\ &+(j-1)\sum_{i=j-1}^{D}w_{j,i+2-j}\rho_{ik}(\ell)+(k-1)\sum_{i=k-1}^{D}w_{k,i+2-k}\rho_{ji}(\ell)\\ &+\delta_{j1}\delta_{\ell''_1,\ell-1}w_1p_R(\ell)+\delta_{j1}\delta_{\ell 0}w_1\Bigg\}\end{aligned}$$

for $i,j\geq 1$. Now assume that $\rho_{jk}(\ell)=\rho_{jk}+r_{jk}\ell^{-1}+O(\ell^{-2})$ and that a similar expansion holds for $\rho_j(\ell)$. Expanding the above recursion in ℓ^{-1} gives

$$\begin{aligned}\rho_{jk}+r_{jk}\ell^{-1}+O(\ell^{-2}) \quad &= \quad \left(1-\frac{w_1+2w_2-w_3}{w_2}\ell^{-1}+O(\ell^{-2})\right)\times\\ &\Bigg\{\left(1+\frac{-w_j-w_k+2w_1-w_2}{w_2}\ell^{-1}\right)\left(\rho_{jk}+r_{jk}\ell^{-1}+O(\ell^{-2})\right)\\ &+\frac{\ell^{-1}}{w_2}\Bigg[(j-1)w_{j,k}\big(\rho_{j+k-2}+O(\ell^{-1})\big) \;+\; (j-1)\sum_{i=j-1}^{D}w_{j,i+2-j}\left(\rho_{ik}+O(\ell^{-1})\right)\\ &+(k-1)\sum_{i=k-1}^{D}w_{k,i+2-k}\left(\rho_{ji}+O(\ell^{-1})\right)\Bigg]\Bigg\}.\end{aligned}$$

This equation is trivially satisfied in zeroth order of ℓ^{-1}. When we go to the next order we get the following equation for the limits of the densities

$$\begin{aligned}\rho_{jk} &= -\frac{w_j + w_k}{w_2}\rho_{jk} + (j-1)\frac{w_{j,k}}{w_2}\rho_{j+k-2} + (j-1)\sum_{i=j-1}^{D}\frac{w_{j,i+2-j}}{w_2}\rho_{ik} \\ &\quad +(k-1)\sum_{i=k-1}^{D}\frac{w_{k,i+2-k}}{w_2}\rho_{ji}. \end{aligned} \tag{8.10}$$

We can also arrive directly at this equation by assuming that for large ℓ an equilibrium with small enough fluctuations is established, and then perform the splitting procedure on this equilibrium. With the same methods, it is possible to derive an equation like (8.10) for the density $\rho_{j_1,j_2,\ldots,j_R}$ of linear paths of length $R-1$ directed towards the root containing vertices of degrees $j_1,\ldots,j_R$. This would allow us to investigate how the correlations fall of with distance R.

Existence of solutions to Equation (8.10) can be established by the Perron–Frobenius argument as in the previous sections. In the following subsections we will find an explicit solution for linear weights and discuss generalizations for non–linear weights. In both cases we compare the results with simulations.

8.2 Solution in the simplest case

When $D = 3$, the splitting weights are linear and the partitioning weights uniform, we know that $\rho_1 = \rho_3 = 2/7$ and $\rho_2 = 3/7$, see Chapter 6. Let $y = w_3/w_2$. Then the solutions to Equation (8.10) are

$$\begin{aligned} \rho_{21} &= \frac{4(3-y)}{7(11-2y)}, & \rho_{31} &= \frac{10}{7(11-2y)}, \\ \rho_{22} &= \frac{4y^2-12y+105}{7(2y+7)(11-2y)}, & \rho_{32} &= \frac{2(-8y^2+18y+63)}{7(2y+7)(11-2y)}, \\ \rho_{23} &= \frac{2(-4y^2+20y+21)}{7(2y+7)(11-2y)}, & \rho_{33} &= \frac{8(3y-14)}{7(2y+7)(2y-11)}. \end{aligned} \tag{8.11}$$

Note that the following sum rules hold for the solutions

$$\begin{aligned} \rho_{21} + \rho_{31} &= \rho_1 &&= 2/7 \\ \rho_{22} + \rho_{32} &= \rho_2 &&= 3/7 \\ \rho_{23} + \rho_{33} &= \rho_3 &&= 2/7, \\ \rho_{21} + \rho_{22} + \rho_{23} &= \rho_2 &&= 3/7 \\ \rho_{31} + \rho_{32} + \rho_{33} &= 2\rho_3 &&= 4/7. \end{aligned} \tag{8.12}$$

These relations show that there are only two independent link densities. We plot ρ_{21} and ρ_{22} in Figure 8.6 and compare to simulations.

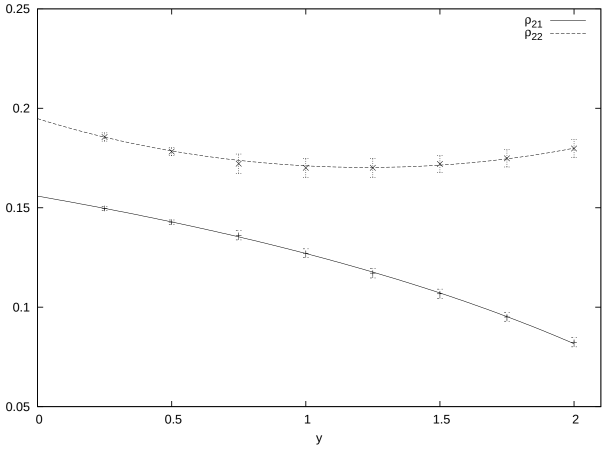


Figure 8.6: Two independent solutions given in (8.11) plotted against $y = w_3/w_2$ and compared to simulations. The two leftmost data points on each line come from simulations of 50 trees with 50000 vertices. The other data points come from simulations of 50 trees with 10000 vertices.

8.3 Results for non–linear weights

We can generalize Equation (8.10) to a mean field equation, valid for arbitrary weights, by replacing w_2, where it appears in a denominator, with w as we did with the equation for vertex degree densities in Chapter 6. For $D = 3$ and uniform partitioning weights the two independent densities ρ_{21} and ρ_{22} are given by

$$\rho_{21} = \frac{1}{3} \frac{(3+\beta)(7\,\alpha-\gamma)}{(2\,\alpha-\beta-1)(3\,\alpha+2\,\beta+\gamma+6)} \tag{8.13}$$

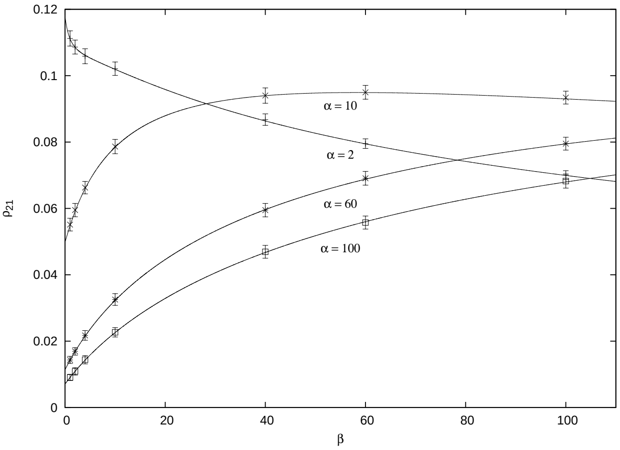


Figure 8.7: A solution for the density ρ_{21} plotted as a function of β for a few values of α. Each datapoint is calculated from simulations of 100 trees with 10000 vertices.

$$\begin{aligned}
\rho_{22} \quad = \quad & \frac{16}{3}\Big(284\,\alpha^2\beta^4\gamma - 177\,\alpha^5\beta\gamma + 3564\,\alpha^3 + 18\,\alpha^6\gamma + 161\alpha\,\beta^5\gamma - 873\,\gamma + 11979\,\alpha^2\beta^3 \\
& -2259\,\alpha^5 - 39\,\alpha^6\beta - 207\,\alpha^5\gamma + 6516\,\alpha^2\beta^4 - 5205\,\alpha^5\beta - 1419\,\alpha^4\beta\gamma + 996\,\alpha\beta^5 \\
& -5994\,\alpha^4 - 892\,\alpha^4\beta^2\gamma + 1543\,\alpha^2\beta^5 - 18\,\alpha^7 - 668\,\alpha^3\beta^4 + 324\,\alpha^2\gamma + 909\,\alpha\beta^3\gamma \\
& -2600\,\alpha^5\beta^2 - 975\,\alpha^3\beta^3 + 222\,\alpha\beta^6 - 1533\,\alpha^3\beta^2\gamma + 10206\,\alpha^2\beta^2 - 11799\,\alpha^4\beta \\
& -5300\,\alpha^4\beta^3 - 1521\,\alpha^3\beta\gamma + 1899\,\alpha^2\beta^2\gamma + 1059\alpha^2\,\beta^3\gamma + 1269\,\alpha^3\beta^2 + 3240\alpha^2\,\beta \\
& +756\,\alpha\beta^3 + 4860\,\alpha^3\beta + 6\,\beta^6\gamma - 11703\,\alpha^4\beta^2 + 1728\alpha^2\,\beta\gamma - 162\,\alpha^3\gamma + 486\alpha\,\beta^2\gamma \\
& +18\,\beta^4\gamma + 1530\,\alpha\beta^4 + 624\alpha\,\beta^4\gamma - 772\,\alpha^3\beta^3\gamma - 9\,\alpha^6 + 24\,\beta^5\gamma\Big)\Big/\Big(\,(3\,\alpha + 2\,\beta + \gamma + 6) \\
& \times\Big(11\,\alpha^2 + 25\,\alpha\beta + 5\,\alpha\gamma + 3\,\beta\gamma + 12\,\alpha + 4\,\beta^2\Big)\,(-\alpha + \gamma)\,(1 - 2\,\alpha + \beta)\,(7\,\alpha + 2\,\beta + \gamma)^2\,\Big)
\end{aligned}$$

where $\alpha = \dfrac{w_2}{w_1}$, $\beta = \dfrac{w_3}{w_1}$ and $\gamma = \sqrt{\alpha\,(\alpha + 24\,\beta + 24)}$. These solutions are compared to simulations in Figures 8.7 and 8.8. The other densities are obtained by using the sum rules (8.12).

8.4 The correlation coefficient

We have calculated the correlation coefficient r in the case of linear splitting weights and $D = 3$. There are two independent parameters which we take to be

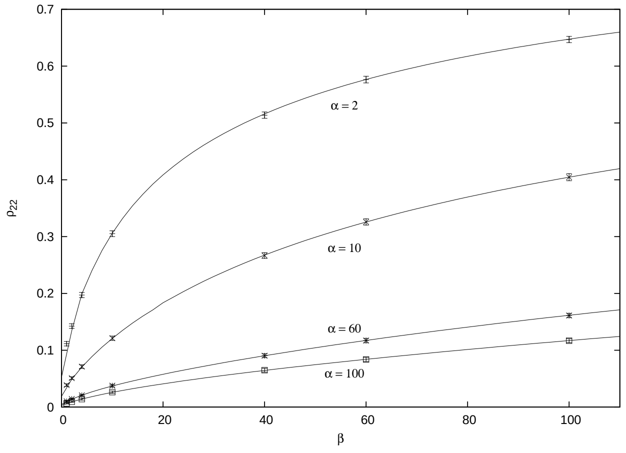


Figure 8.8: A solution for the density ρ_{22} plotted as a function of β for a few values of α. Each data point is calculated from simulations of 100 trees with 10000 vertices.

$y = w_3/w_2 \in [0,2]$ and $z = w_{2,2}/w_2 \in [0,1]$. We find that

$$r = -\frac{18\,z^3 - 42\,z^2y - 27\,z^2 - 4\,zy^2 + 132\,zy - 96\,y + 4\,y^2}{(3\,z-5)\,(3\,z+2\,y-12)\,(6\,z-2\,y-9)} \tag{8.14}$$

and we plot r in Fig. 8.9 as a function of y and z.

The trees show disassortative mixing except when $y = z = 0$ in which case $r = 0$.

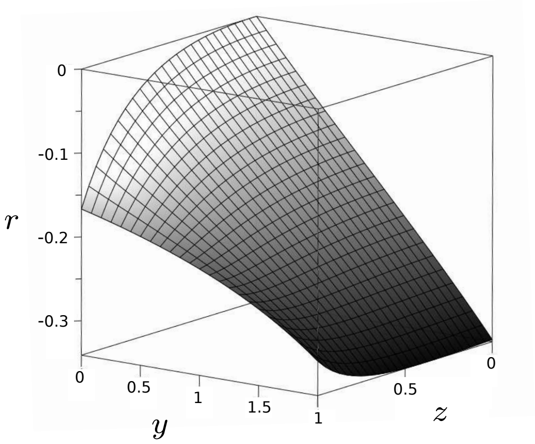


Figure 8.9: The correlation coefficient r plotted as a function of $y = w_3/w_2$ and $z = w_{2,2}/w_2$ in the case of $D = 3$ and linear splitting weights.

This corresponds to the special case of the preferential attachment model. We have checked that for $D = 4$, the preferential attachment model again has $r = 0$. It would be interesting to check whether this holds for all values of D. In practise, the question of assortative mixing is more relevant for large D and it would be desirable to obtain such general results.

9

Relation to other models of random trees

In this chapter we discuss how the vertex splitting model is related to other models of random trees. It has already been noted that the preferential attachment model is a special case which corresponds to choosing the only nonzero weights to be $w_{k,1} = w_{1,k},\ 2 \leq k \leq D$. In the following sections we introduce the alpha model and its generalization, the alpha–gamma model. We demonstrate how they arise as a certain limiting case of the vertex splitting model and discuss how they are connected to the tree models from Part I. We analyse some properties of these models and provide results which support some of the scaling assumptions from the previous chapters.

9.1 The alpha model

The alpha model is a one parameter model of growing, rooted, binary trees which was introduced by D. Ford in [43] as a model of phylogenetic trees. Below, we will state the growth rules of the alpha model and explain how it is related to both the vertex splitting model and the generic phases of the models from Part I. We prove that the finite volume measures generated by the growth rules converge to a measure on infinite graphs, and we calculate the annealed Hausdorff dimension with respect to the infinite volume measure as defined in Equation (1.21). It turns out that the annealed Hausdorff dimension agrees with the values obtained by Equation (7.44).

The root and vertices of degree 3 will be referred to as *internal vertices* and vertices of degree 1 (besides the root) will be referred to as *leaves*. Denote the set of rooted, planar trees on n leaves by T_n and in a tree τ denote the number of leaves by $l(\tau)$. The model is defined by probability distributions $\pi_{\alpha,n}$ on T_n, for $n \geq 1$ and $0 \leq \alpha \leq 1$, constructed in the following recursive way by a growth rule.

- Assign probability one to the unique trees in T_1 and T_2.
- Given $\pi_{\alpha,n}$ for some $n \geq 2$, $\pi_{\alpha,n+1}$ is generated by first selecting a tree $\tau \in T_n$ according to $\pi_{\alpha,n}$.
- Next an individual edge (a,b) is selected from τ with probability $\alpha/(n-\alpha)$ if a and b are internal vertices and with probability $(1-\alpha)/(n-\alpha)$ if one is an internal vertex and the other a leaf.
- The edge (a,b) is removed from τ and two new vertices c and d are introduced along with the edges (a,c), (c,b) and (c,d). Equal probability is assigned to left and right branching of the new edge (c,d).

One can think about this procedure as grafting a new edge to an existing edge in τ, see Fig. 9.1.

The alpha model is equivalent to a slightly modified version of the vertex splitting model with the following choice of weights. Consider the case $D = 3$ and choose

$$w_1 = 1 - \frac{3\alpha}{2}, \qquad w_{3,2} = \frac{\alpha}{2} \qquad \text{and} \quad w_{3,1} = \infty. \tag{9.1}$$

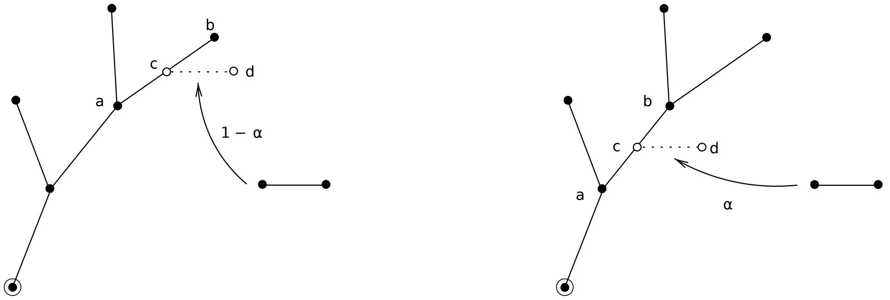


Figure 9.1: The grafting process. Left: The edge (a,b) is selected with probability weight $1-\alpha$. Right: The edge (a,b) is selected with probability weight α. The selected edge is removed, two new vertices c and d and three new edges are added as shown in the figure. The root is indicated by circled vertex.

We modify the model by assigning a weight $\frac{\alpha}{2}$ (instead of w_1) to splitting the root. As the tree grows large it becomes very improbable to split the root and therefore we expect this to play no role in the calculation of asymptotic properties such as the vertex degree distribution and the Hausdorff dimension. The relation between the grafting process of the alpha model and the above splitting operations is described in Fig. 9.2. Note from Equation (9.1) that for $2/3 < \alpha \leq 1$, w_1 becomes negative which means that the vertex splitting description breaks down. However, even though w_1 is a negative weight, the total probability of the process which involves w_1 splittings (see Fig. 9.2, top) is still nonnegative since $w_{2,1} + w_{3,2} > 0$.

The alpha model is also related to the models of generic caterpillars and generic trees, which were discussed in Part I. The case $\alpha = 1$ corresponds to the generic caterpillars obtained by choosing the branching weights $w_1 = w_3 = 1$ and $w_2 = 0$. In the case $\alpha = 1/2$ the growth process does not distinguish between leaves and internal edges and therefore it generates the uniform measure on binary trees. This corresponds to generic trees defined by the branching weights $w_1 = w_3 = 1$ and $w_2 = 0$. We know from Part I that in the generic phase of both of these models, the finite volume measures converge to measures concentrated on the set of trees with precisely one infinite spine having finite outgrowths. It is therefore reasonable to conjecture that the same applies in the alpha model, at least for $1/2 \leq \alpha \leq 1$. In the next section we will prove that this is indeed true, and holds for $0 < \alpha \leq 1$.

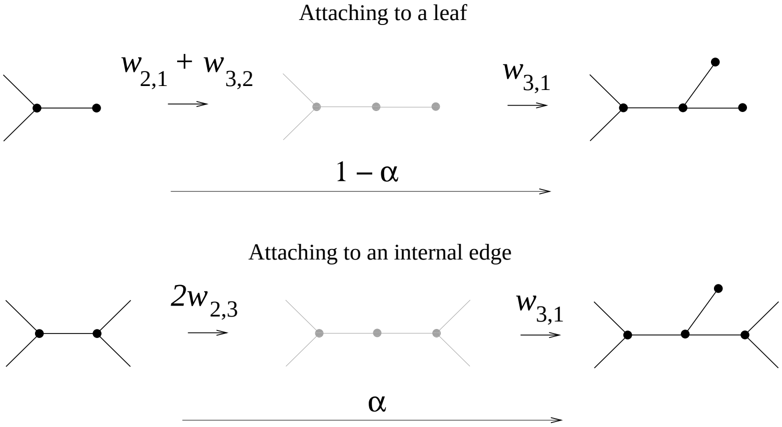


Figure 9.2: The relation between the alpha model and the vertex splitting model explained. Since $w_{1,3} = \infty$ a vertex of degree 2 splits immediately, with probability one, to a vertex of degree 1 and a vertex of degree 3 and the intermediate state is not realized.

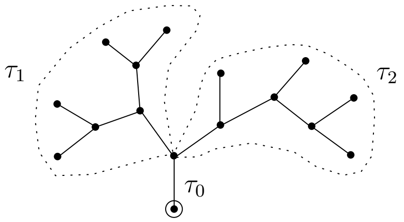


Figure 9.3: An example of a tree τ_0 which has a root indicated by a circled vertex. The tree τ_0 branches at the nearest neighbour of the root to two subtrees, τ_1 to the left and τ_2 to the right as is indicated by the dotted lines.

9.1.1 Convergence of the finite volume measures

The alpha model has a property called *Markovian self-similarity* [43] which is essential in the inductive proof of the theorem in this section. Markovian self-similarity means that there exists a function $q_\alpha(\cdot,\cdot)$, which is called the *first split distribution*, such that for every finite tree τ_0 which branches at the nearest neighbour of the root to a left tree τ_1 and a right tree τ_2 (see Fig. 9.3) the following holds

$$\pi_{\alpha,l(\tau_0)}(\tau_0) = q_\alpha(l(\tau_1), l(\tau_2))\pi_{\alpha,l(\tau_1)}(\tau_1)\pi_{\alpha,l(\tau_2)}(\tau_2). \tag{9.2}$$

In words, this says that $q_\alpha(n_1, n_2)$ is the probability of a tree branching to subtrees of sizes n_1 and n_2. Furthermore, given that the subtrees are of these sizes they are distributed independently by π_{α,n_1} and π_{α,n_2}. The function q_α is explicitly known [43] and is given by

$$q_\alpha(n_1, n_2) = \frac{n!\Gamma_\alpha(n_1)\Gamma_\alpha(n_2)}{n_1!n_2!\Gamma_\alpha(n)}\left(\frac{\alpha}{2} + \frac{(1-2\alpha)n_1n_2}{n(n-1)}\right) \tag{9.3}$$

where $n = n_1 + n_2$,

$$\Gamma_\alpha(n) = (n-1-\alpha)(n-2-\alpha)\cdots(2-\alpha)(1-\alpha), \quad \text{and} \quad \Gamma_\alpha(1) = 1. \tag{9.4}$$

Using this property we can prove the following theorem.

Theorem 9.1.1 *Let $0 < \alpha \leq 1$. The measures $\pi_{\alpha,n}$, viewed as probability measures on Γ, converge weakly, as $n \longrightarrow \infty$, with respect to the metric d defined in (3.4)*[1]*, to a probability measure π_α on the set of infinite trees. The measure π_α is concentrated*

---

[1] Since the degree of vertices is ≤ 3 it is equivalent to work with the standard metric used in [65]. It is defined as in (3.4), replacing L_R with B_R.

on the set of trees with exactly one infinite rooted spine having finite outgrowths which are independently distributed by

$$\mu_\alpha(\tau) = \frac{\alpha\Gamma_\alpha(l(\tau))}{l(\tau)!}\pi_{\alpha,l(\tau)}(\tau). \tag{9.5}$$

The probabilities of right and left branching of outgrowths are equal.

Proof Let $T^{(R)}$ be the set of rooted, binary trees of height R. To prove the existence of π_α, it is sufficient to show that for any $R \geq 1$ and any $\tau_0 \in T^{(R)}$ the sequence

$$\pi_{\alpha,n}(\{\tau | B_R(\tau) = \tau_0\}) \equiv \pi^{(R)}_{\alpha,n}(\tau_0) \tag{9.6}$$

converges to a limit $\pi^{(R)}_\alpha(\tau_0)$ as $n \longrightarrow \infty$, c.f. Section 1.3.2. We show this by induction on R. For $R = 1$ this is trivial since $B_1(\tau) \in T^{(1)}$ for all τ. Now assume that for some R and all $\tau \in T^{(R)}$, $\pi^{(R)}_{\alpha,n}(\tau)$ converges as $n \longrightarrow \infty$. Choose a tree $\tau_0 \in T^{(R+1)}$ and without loss of generality, assume it branches at the nearest neighbour of the root to a left tree $\tau_1 \in T^{(R)}$ and a right tree $\tau_2 \in T^{(S)}$ (see Fig. 9.3) where $S \leq R$. Then, by Equation (9.3),

$$\begin{aligned}\pi^{(R+1)}_{\alpha,n}(\tau_0) &= \sum_{n_1+n_2=n} q_\alpha(n_1,n_2)\pi^{(R)}_{\alpha,n_1}(\tau_1)\pi^{(R)}_{\alpha,n_2}(\tau_2) \\ &= \frac{n!}{\Gamma_\alpha(n)}\Big(\frac{\alpha}{2}\sum_{n_1+n_2=n}\frac{\Gamma_\alpha(n_1)\Gamma_\alpha(n_2)}{n_1!n_2!}\pi^{(R)}_{\alpha,n_1}(\tau_1)\pi^{(R)}_{\alpha,n_2}(\tau_2) \\ &\quad + \frac{1-2\alpha}{n(n-1)}\sum_{n_1+n_2=n}\frac{\Gamma_\alpha(n_1)\Gamma_\alpha(n_2)}{(n_1-1)!(n_2-1)!}\pi^{(R)}_{\alpha,n_1}(\tau_1)\pi^{(R)}_{\alpha,n_2}(\tau_2)\Big).\end{aligned} \tag{9.7}$$

If $S < R$ then $\pi^{(R)}_{\alpha,n_2}(\tau_2) = 0$ when $n_2 > l(\tau_2)$ and it is obvious from the induction hypothesis that $\pi^{(R+1)}_{\alpha,n}(\tau_0)$ converges. Therefore assume that $S = R$.

Note that in (9.7) it always holds that either $n_1 \leq n-1$ and $n_2 \leq n$ or $n_2 \leq n-1$ and $n_1 \leq n$. Therefore we have the upper bound

$$\pi^{(R+1)}_{\alpha,n}(\tau_0) \leq \frac{n!}{\Gamma_\alpha(n)}\sum_{n_1+n_2=n}\frac{\Gamma_\alpha(n_1)\Gamma_\alpha(n_2)}{n_1!n_2!}.$$

Terms in the sums in (9.7) for which $n_1 \geq \frac{n}{2}$ and $n_2 > A$ or $n_2 \geq \frac{n}{2}$ and $n_1 > A$ where $A > 1$ is some constant are therefore bounded from above by

$$\begin{aligned}\frac{2n!}{\Gamma_\alpha(n)} \sum_{\substack{n_1+n_2=n\\ n_1\geq n/2, n_2>A}} \frac{\Gamma_\alpha(n_1)\Gamma_\alpha(n_2)}{n_1!n_2!} &\leq \frac{2n!\Gamma_\alpha([n/2])}{\Gamma_\alpha(n)[n/2]!} \sum_{n_2=A}^{\infty} \frac{\Gamma_\alpha(n_2)}{n_2!} \\ &\leq C \sum_{n_2=A}^{\infty} \frac{\Gamma_\alpha(n_2)}{n_2!} \quad \xrightarrow[A\to\infty]{} 0 \end{aligned} \tag{9.8}$$

where C is a constant. The remaining contribution to (9.7) is from terms where $n_1 \geq \frac{n}{2}$ and $n_2 < A$ or $n_2 \geq \frac{n}{2}$ and $n_1 < A$. Notice that the second term in that contribution to (9.7) will be of one order lower in n than the first term. Therefore it is enough to show that the first term converges as $n \longrightarrow \infty$ since then the second term clearly converges to zero. The contribution to the first term is

$$\begin{aligned} &\frac{n!}{\Gamma_\alpha(n)} \frac{\alpha}{2} \sum_{i=1}^{2} \sum_{\substack{n_1+n_2=n\\ n_j\leq A, j\neq i}} \frac{\Gamma_\alpha(n_1)\Gamma_\alpha(n_2)}{n_1!n_2!} \pi_{\alpha,n_1}^{(R)}(\tau_1)\pi_{\alpha,n_2}^{(R)}(\tau_2) \\ &\xrightarrow[n\to\infty]{} \frac{1}{2} \sum_{\substack{i=1\\ j\neq i}}^{2} \pi_\alpha^{(R)}(\tau_i) \sum_{m=1}^{A} \frac{\alpha\Gamma_\alpha(m)}{m!} \pi_{\alpha,m}^{(R)}(\tau_j) \\ &\xrightarrow[A\to\infty]{} \frac{1}{2} \sum_{\substack{i=1\\ j\neq i}}^{2} \pi_\alpha^{(R)}(\tau_i) \sum_{m=1}^{\infty} \frac{\alpha\Gamma_\alpha(m)}{m!} \pi_{\alpha,m}^{(R)}(\tau_j). \end{aligned} \tag{9.9}$$

In the first step we used the induction hypothesis. This is the limit of $\pi_{\alpha,n}^{(R+1)}(\tau_0)$ as $n \longrightarrow \infty$. The fact that the sum in (9.8) converges to zero as $A \longrightarrow \infty$ proves that the measure is concentrated on the set of trees with exactly one infinite spine. The last sum in (9.9) shows that the distribution of the finite outgrowths is given by μ_α.

□

9.1.2 The annealed Hausdorff dimension

In [43], Ford proves that for $0 < \alpha \leq 1$, the expected distance of a random leaf in a tree chosen from the alpha model with n leaves is $O(n^\alpha)$. This means that the Hausdorff dimension, as defined in (1.25), of the alpha model is $d_H = 1/\alpha$. Indeed, by plugging the weights (9.1) into Equation (7.44) we find that

$$d_H = \frac{w_{2,1} + 3w_{3,2}}{2w_{3,2}} = 1/\alpha \tag{9.10}$$

which supports the validity of the formula (7.44). In this section we use the results of Theorem 9.1.1 to prove that the annealed Hausdorff dimension with respect to the measure π_α equals $1/\alpha$ which demonstrates that the two definitions of the Hausdorff dimension, (1.21) and (1.25), agree for the alpha model.

Theorem 9.1.2 *For $0 < \alpha \leq 1$, the annealed Hausdorff dimension with respect to π_α is*

$$\bar{d}_H = 1/\alpha. \tag{9.11}$$

Proof We need to analyse the large R behaviour of $\langle |B_R| \rangle_{\pi_\alpha}$. Let τ be a finite outgrowth from the spine and take the unique vertex common to τ and the spine to be the root of τ. It is clearly sufficient to show that

$$\langle |B_R(\tau)| \rangle_{\mu_\alpha} \sim R^{1/\alpha - 1} \tag{9.12}$$

as $R \longrightarrow \infty$ since the outgrowths from the spine are i.i.d. Define the probability generating function

$$f_R(z) = \sum_{i=1}^{2^{R-1}} \mu_\alpha(\{\tau \mid |B_R(\tau)| = 2i-1\}) z^{2i-1}. \tag{9.13}$$

Consider the contribution to (9.13) from trees on n leaves and define

$$A_{n,R}(z) = \sum_{i=1}^{2^{R-1}} \alpha \frac{\Gamma_\alpha(n)}{n!} \pi_{\alpha,n}(\tau \mid |B_R| = 2i-1) z^{2i-1} \tag{9.14}$$

and the corresponding generating function

$$G_R(z,\zeta) = \sum_{n=1}^{\infty} A_{n,R}(z) \zeta^n. \tag{9.15}$$

It then follows that

$$\langle |B_R| \rangle_{\mu_\alpha} = \partial_z f_R(z)|_{z=1} = \partial_z G_R(z,1)|_{z=1}. \tag{9.16}$$

Using the Markovian self–similarity property of the alpha model we can derive the following recursion

$$A_{1,R}(z) = \alpha z \tag{9.17}$$

and for $n \geq 2$

$$\begin{aligned} A_{n,R}(z) &= \delta_{R,1} z\alpha \frac{\Gamma_\alpha(n)}{n!} + z \sum_{i=2}^{2^{R-1}} \alpha \frac{\Gamma_\alpha(n)}{n!} \sum_{n_1+n_2=n} q_\alpha(n_1,n_2) \\ &\times \sum_{\substack{j+k=i \\ 1\leq j,k\leq 2^{R-2}}} \pi_{\alpha,n_1}(|B_{R-1}|=2j-1)\pi_{\alpha,n_2}(|B_{R-1}|=2k-1)z^{2j-1}z^{2k-1} \\ &= \delta_{R,1} z\alpha \frac{\Gamma_\alpha(n)}{n!} + z\Big(\frac{1}{2}\sum_{n_1+n_2=n} A_{n_1,R-1}(z)A_{n_2,R-1}(z) \\ &+ \frac{1-2\alpha}{\alpha}\frac{1}{n(n-1)}\sum_{n_1+n_2=n} n_1 A_{n_1,R-1} n_2 A_{n_2,R-1}\Big). \end{aligned}$$

Writing the above recursion in terms of the generating function G_R one finds

$$G_1(z,\zeta) = z\sum_{n=1}^{\infty} \alpha \frac{\Gamma_\alpha(n)}{n!}\zeta^n = z\left(1-(1-\zeta)^\alpha\right) \tag{9.18}$$

and for $R > 1$

$$G_R(z,\zeta) = z\left(\alpha\zeta + \frac{1}{2}\left(G_{R-1}(z,\zeta)\right)^2 + \frac{1-2\alpha}{\alpha}\int_0^\zeta \int_0^{\zeta'} \left(\partial_{\zeta''} G_{R-1}(z,\zeta'')\right)^2 d\zeta'' d\zeta'\right). \tag{9.19}$$

It is straightforward to verify that for all R

$$G_R(1,\zeta) = 1-(1-\zeta)^\alpha. \tag{9.20}$$

Define $V_R(\zeta) = \partial_z G_R(z,\zeta)|_{z=1}$. Differentiating the recursion (9.19) with respect to z and putting $z = 1$ one then gets

$$V_1(\zeta) = 1-(1-\zeta)^\alpha \tag{9.21}$$

and for $R > 1$

$$V_R(\zeta) = (1-(1-\zeta)^\alpha)\left(1+V_{R-1}(\zeta)\right) + 2(1-2\alpha)\int_0^\zeta \int_0^{\zeta'} (1-\zeta'')^{\alpha-1}\partial_{\zeta''} V_{R-1}(\zeta'') d\zeta'' d\zeta'. \tag{9.22}$$

Differentiating (9.22) twice one finds that

$$V''_{R+1} = \alpha(1-\alpha)(1-\zeta)^{\alpha-2}(1+V_R) + 2(1-\alpha)(1-\zeta)^{\alpha-1}V'_R + (1-(1-\zeta)^\alpha)V''_R \tag{9.23}$$

and the initial conditions

$$V_R(0) = 0 \qquad \text{and} \qquad V_R'(0) = \alpha \qquad \text{for all } R \tag{9.24}$$

follow from (9.21) and (9.22). Define the generating function

$$Q_x(\zeta) = \sum_{R=1}^{\infty} V_R(\zeta) x^R. \tag{9.25}$$

From (9.23) we get the differential equation

$$\left(\left(\frac{1}{x} - 1\right)(1-\zeta)^{-\alpha} + 1\right)(1-\zeta)^2 Q_x'' - 2(1-\alpha)(1-\zeta)Q_x' - \alpha(1-\alpha)Q_x = \frac{\alpha(1-\alpha)}{1-x} \tag{9.26}$$

with initial conditions

$$Q_x(0) = 0 \qquad \text{and} \qquad Q_x'(0) = \frac{\alpha x}{1-x}. \tag{9.27}$$

Let $y(\zeta) = -(1-\zeta)^{\alpha}(\frac{1}{x} - 1)^{-1}$ and define $P_x(y(\zeta)) = Q_x(\zeta)$. Then $P_x(y)$ satisfies the differential equation

$$\frac{\alpha}{1-\alpha} y(y-1)P_x'' + (y+1)P_x' - P_x = \frac{1}{1-x} \tag{9.28}$$

with initial conditions

$$P_x\left(\frac{x}{x-1}\right) = 0 \quad \text{and} \qquad P_x'\left(\frac{x}{x-1}\right) = 1 \tag{9.29}$$

which is equivalent to (9.26) and (9.27). Equation (9.28) is an inhomogeneous, hypergeometric differential equation which has the general solution

$$P_x(y) = C_1(x)\,(y+1) + C_2(x) y^{1/\alpha} F\left(\frac{2-\alpha}{\alpha}, \frac{1-\alpha}{\alpha}, \frac{\alpha+1}{\alpha}; y\right) - \frac{1}{1-x}, \tag{9.30}$$

where F is a hypergeometric function and $C_1(x)$, $C_2(x)$ are functions independent of y, see e.g. [25, Chapter 9, §10]. Thus

$$Q_x(1) = P_x(0) = C_1(x) - \frac{1}{1-x}. \tag{9.31}$$

We use the initial conditions on P_x to find $C_1(x)$ and get

$$\begin{aligned}
C_1(x) &= \frac{1}{(\alpha x - 2x + 1)F\left(\frac{2-\alpha}{\alpha}, 2, \frac{1+\alpha}{\alpha}; 1-x\right) + (2-\alpha)x(2x-1)F\left(\frac{2}{\alpha}, 2, \frac{1+\alpha}{\alpha}; 1-x\right)} \\
&\times \left(\frac{2\Gamma\left(\frac{1}{\alpha}\right)^2}{\Gamma\left(\frac{2-\alpha}{\alpha}\right)} x^{\frac{2\alpha-1}{\alpha}} (1-x)^{-1/\alpha} + (\alpha x + 1)F\left(\frac{2-\alpha}{\alpha}, 2, \frac{1+\alpha}{\alpha}; 1-x\right)\right. \\
&\qquad \left. -(2-\alpha)xF\left(\frac{2}{\alpha}, 2, \frac{1+\alpha}{\alpha}; 1-x\right)\right) \\
&= \frac{2\Gamma\left(\frac{1}{\alpha}\right)^2}{\Gamma\left(\frac{2-\alpha}{\alpha}\right)} (1-x)^{-1/\alpha} \left(1 + O((1-x)^{1/\alpha})\right).
\end{aligned}$$

This shows that

$$\langle |B_R| \rangle_{\mu_\alpha} = \partial_z G_R(z,1)|_{z=1} = V_R(1) \sim R^{1/\alpha - 1} \tag{9.32}$$

and thus

$$\bar{d}_H = 1/\alpha. \tag{9.33}$$

□

9.2 The alpha–gamma model

A generalization of the alpha model to a two parameter model of trees was introduced in [28] in the so called *alpha–gamma model*. A new step was added to the growth process, allowing links to be attached to vertices and thereby increasing their degrees. The parameters of the models are positive numbers α and γ obeying $0 \leq \gamma \leq \alpha \leq 1$ and the growth rules are the following. Graft a new edge to either side of an internal edge with probability weight γ, to either side of a leaf with probability weight $1-\alpha$ and anywhere to a vertex of degree $k \geq 3$ with probability weight $(k-2)\alpha - \gamma$. This growth process generates a probability measure on trees with n leaves which we will denote by $\pi_{\alpha,\gamma,n}$. When $\alpha = \gamma$ we recover the alpha model i.e. $\pi_{\alpha,\alpha,n} = \pi_{\alpha,n}$. The alpha–gamma model is Markovian self–similar [28] and we denote the first split distribution by $q_{\alpha,\gamma,n}$.

The above growth rules can be obtained from the rules of the vertex splitting model as in the case of the alpha model. The nonzero weights are then

$$w_1 = 1 - \frac{3\alpha}{2}, \qquad w_{3,1} = \infty, \tag{9.34}$$

$$w_{k,2} \quad = \quad \frac{\alpha}{2} \quad \text{and} \quad w_{k+1,1} = \frac{2((k-2)\alpha-\gamma)}{k} \quad \text{for } k \geq 3. \tag{9.35}$$

In [28] it is remarked that for $\gamma = 1-\alpha$ the continuum limit of the alpha–gamma model is equivalent to the stable tree of Duquesne and Le Gall [34] with parameter $1/\alpha$. The stable trees with parameter $1/\alpha$ can be viewed as the continuum limit of a size conditioned critical Galton–Watson process with offspring probabilities defined by (3.22) with $\beta = 1 + 1/\alpha$ and $w_1 = w_c$.

The alpha–gamma model also provides a connection between the vertex splitting model and the critical line in the equilibrium statistical mechanical model of caterpillars from Part I. As noted in [28], the choice $\alpha = 1$ and $0 < \gamma \leq 1$ yields a model of growing caterpillars with a first split distribution

$$q_{1,\gamma}(n_1,\ldots,n_k) = \begin{cases} \gamma\Gamma_\gamma(k-1)/(k-1)! & \text{if } 2 \leq k \leq n-1 \text{ and } n_i = n-k+1, \\ & n_j = 1,\ i \neq j, \text{ for some } i, \\ \gamma\Gamma_\gamma(n-1)/(n-2)! & \text{if } k = n \text{ and } (n_1,\ldots,n_k) = (1,\ldots,1), \\ 0 & \text{otherwise.} \end{cases} \tag{9.36}$$

It is straightforward to prove that $\pi_{1,\gamma,n}$ converges weakly as $n \longrightarrow \infty$ to a measure $\pi_{1,\gamma}$ which is concentrated on the set of infinitely long caterpillars and that the degrees k on the spine are independently distributed by

$$\gamma\Gamma_\gamma(k-2)/(k-2)!. \tag{9.37}$$

The measure $\pi_{1,\gamma}$ is the same measure as is obtained for the equilibrium statistical mechanical model of caterpillars with the branching weights (2.41), and $\beta = \gamma + 2$. Theorems 2.4.2 and 2.4.3 therefore apply and we find that the Hausdorff dimension of the alpha-gamma model with $\alpha = 1$ is $\pi_{1,\gamma}$–almost surely $d_H = 1/\gamma$ and the spectral dimension is $\pi_{1,\gamma}$–almost surely $d_s = 2/(1+\gamma)$.

We conclude this chapter by summarizing in a diagram the relation between the models presented in the thesis.

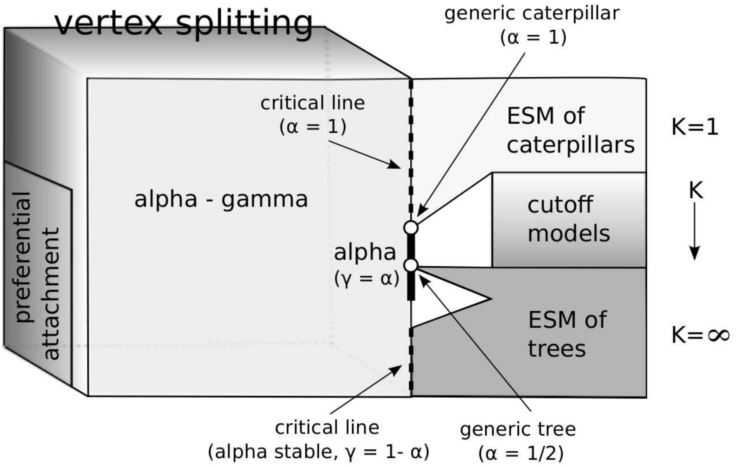


Figure 9.4: Relation between the models presented in the thesis. The vertex splitting model is represented by a cube. The left side of the cube contains the preferential attachment model. The front side of the cube is the limiting case of the alpha–gamma model, and the special case of the alpha-model is represented by a thick line on the front–right edge. The models of Part I intersect the front–right edge in various places as indicated. ESM stands for equilibrium statistical mechanics and “cutoff models” refers to the models discussed in Section 2.5 where K is the maximum vertex degree in the outgrowths .

10

Conclusion

In the second part of the thesis we introduced a new model of growing random trees, referred to as the vertex splitting model. We analyzed some properties of large trees such as the vertex degree distribution, correlations between neighbouring vertices and the Hausdorff dimension. Rigorous results were presented in the case of linear splitting weights $w_i = ai + b$ and the case of more general weights was studied by a mean field assumption which was supported by simulations. It would be desirable to extend the rigorous calculations for linear weights to the more general case and thereby confirm the observed mean field behaviour.

The study of the degree distribution involved proving convergence of the expectation value of the relative number of vertices of a given degree. It is possible to strengthen the notion of convergence by showing that the vertex degree densities converge almost surely to their limits. This can presumably be done using results on generalized Pólya urn models as is done in the case of random recursive trees (preferential attachment) in [49].

It is an interesting problem to establish weak convergence of the finite volume measures ν_ℓ, generated by the vertex splitting procedure, to a measure on infinite trees. This was done in Section 9.1 for the special case of the alpha model. It was shown that the infinite volume measure is concentrated on the set of trees consisting of exactly one infinite spine with outgrowths which are finite and i.i.d. Similar results are expected to hold in the alpha–gamma model and a natural next step would be to examine this in detail. The proof of Theorem 9.1.1 relied heavily on the Markovian self–similarity property of the alpha model, which does not seem to be present in general in the vertex splitting model. Therefore, a different approach is required in the vertex splitting model.

If the convergence of the measures is established, one can study properties of the infinite volume measure, such as the Hausdorff dimension and the spectral dimension. In Theorem 9.1.2, the annealed Hausdorff dimension of the alpha model was shown to be $\bar{d}_H = 1/\alpha$. In the case of the alpha–gamma model with $\alpha = 1$ it was shown, using a connection to the caterpillar model from Part I, that $d_H = 1/\gamma$ and $d_s = 2/(1+\gamma)$ almost surely. In this case, Equation (1.23) relating d_H and d_s holds and it would be interesting to examine whether the relation holds in general in the vertex splitting model. A first approach is to use Equation (7.28), for the Hausdorff dimension, and compare it to numerical calculations of the spectral dimension.

Another notion of convergence of graphs is the so–called *continuum limit* or *scaling limit*, obtained by shrinking the edges of a graph while increasing their number. More precisely, a graph G_N of volume N is viewed as a metric space with the graph metric d_{G_N}. Then, for a suitable constant γ, an almost sure convergence of $(G_N, N^{-\gamma} d_{G_N})$ to a metric space $(\mathcal{G}, d)$ is established in the Gromov–Hausdorff sense [44]. The study of convergence in this approach and properties of the limiting objects has been an active area of research in the past two decades, boosted by Aldous' definition of the continuum random tree in 1991 [4]. Since then, much work has been done on trees and planar maps. More details may be found in [55] and the references therein.

In the special case of the alpha model and the alpha–gamma model, the continuum limit has been constructed ([28,45] respectively) in the context of fragmentation processes [16]. It is shown that growth rules of the trees are in one-to-one correspondence with dislocation measures of homogeneous fragmentation processes. An interesting question is whether the same applies in the vertex splitting model. A promising tool to answer this question is the volume distribution function $p_k(n_1, \ldots, n_k)$. A positive answer would determine whether the vertex splitting model falls into the already known categories of dislocation measures for self-similar growing trees or if new classes would be discovered. The latter result would indicate a richer universality class structure of the vertex splitting model which includes other previously studied models as special cases.

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