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Random Brushes and Non-Generic Trees

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Contents

Abstract

In the first part of the thesis we prove inequalities between generating functions for return probabilities of random walks on bundled structures. Bundled structures are constructed by attaching graphs called fibers to a single graph called base, by identifying exactly one vertex of each fiber to exactly one vertex of the base. We apply the inequalities to a class of random bundled structures, called random brushes, where the base is $\mathbb{Z}^d$ viewed as a graph and the fibers are linear graphs of random lengths. Thereby we find that for $d = 2$ all random brushes have spectral dimension $d_s = 2$. For $d = 3$ we have $\frac{5}{2} \leq d_s \leq 3$ and for $d \geq 4$ we have $3 \leq d_s \leq d$.

In the second part we study non-generic random trees. They can be either critical or subcritical. We show that critical trees resemble generic trees in some cases and argue that in other cases their critical exponents can be model dependent. We conjecture that in the subcritical case there is a limiting probability measure supported on trees with exactly one vertex of infinite order. We show that the corresponding random trees have Hausdorff dimension $d_H = \infty$ and spectral dimension $d_s \geq 2$ with a model dependent upper bound.

Ágrip (in Icelandic)

Í fyrri hluta ritgerðar sönnum við ójöfnur milli framleiðandi falla fyrir endurkomulíkur slembiganga á ákveðnum tegundum neta sem við köllum útvaxtanet. Þau eru búin þannig til að byrjað er með net sem kallast grunnnet og á það eru hengd net sem kallast útvextir, með því að samsama nákvæmlega einn hnútpunkt í hverjum útvexti við nákvæmlega einn hnútpunkt á grunnnetinu. Ójöfnunum er beitt á sérstök útvaxtanet sem kallast slembiburstar þar sem grunnnetið er $\mathbb{Z}^d$ og útvextirnir eru línuleg net af handahófskenndri lengd. Með því er sýnt að í tilfellinu $d = 2$ hafa allir slembiburstar litrófsvíddina $d_s = 2$. Þegar $d = 3$ gildir $\frac{5}{2} \leq d_s \leq 3$ og þegar $d \geq 4$ gildir $3 \leq d_s \leq d$.

Í síðari hlutanum skoðum við sérstæð tré en þau skiptast í krítísk og undirkrítísk tré. Við sýnum að í sumum tilfellum líkjast krítísk tré almennum trjám og færum rök fyrir því að í öðrum tilfellum geti þau verið háð líkani. Við getum okkur til um markgildi líkindamáls á undirkrítísk tré þar sem tré með nákvæmlega einn hnútpunkt af óendanlegu stigi fást með líkunum einn. Við sönnum að slík slembitré hafa Hausdorffvídd $d_H = \infty$ og litrófsvídd $d_s \geq 2$ með efri mörk sem háð eru líkani.

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1

Introduction

The generic structure of random geometrical objects is of interest in many branches of physics ranging from condensed matter physics to quantum gravity, see e.g. [1] and [2]. An interesting problem is to define and study dimensions of random geometries. There are many possible definitions of dimensions which agree on the lattice $\mathbb{Z}^d$ viewed as a graph and on smooth manifolds, but they can differ on general graphs and random geometries. One way to define a notion of dimension is to study random walk or diffusion on the geometry. The *spectral dimension* is defined to be d_s if the probability that a random walker returns to its starting point, averaged over the random geometries, behaves as $t^{-d_s/2}$ for large number of steps t. It is equivalently defined if the averaged heat kernel at coinciding points viewed as a function of time has this behavior at large time t. The spectral dimension was first introduced by Alexander and Orbach in [3].

The spectral dimension has been studied analytically for certain classes of random trees in [4–7]. It is convenient to study random walk on trees since trees have no loops and therefore a general walk can be cut into separate walks on smaller trees. This gives recurrence relations which make explicit calculations easier. In [8] the spectral dimension of so called bundled structures is studied. They consist of a single graph called base and a collection of graphs called fibers. The fibers are attached to the base by identifying exactly one vertex of each fiber to a vertex of the base. A random walk can be separated into walks on the base and walks on the fibers in the same way as for trees. Diffusion properties of the whole graph can then be deduced from diffusion properties of the base and the fibers.

When graphs can not be cut into pieces like the trees and the bundled structures it becomes more difficult to do analytical calculations. This is for example the case for triangulations in quantum gravity. The spectral dimension of triangulations has been studied numerically in recent years in [9–13].

Another notion of a dimension comes from looking at the growth of the volume of a ball of size R, denoted $B(R)$, averaged over the random geometry, as R grows large. In $\mathbb{R}^d$ for example, we know that the volume of a ball of radius R grows as R^d. The *Hausdorff dimension* is defined to be d_H if $\langle B(R)\rangle \sim R^{d_H}$ as $R \to \infty$ where $\langle\cdot\rangle$ denotes average over the random geometry. The Hausdorff and spectral dimension do not agree in general but it is not well understood which properties of graphs make them differ.

In Chapter 2 we study the spectral dimension of so called random brushes. They are bundled structures with a base $\mathbb{Z}^d$ viewed as a graph and the fibers are linear graphs of random lengths. This is a generalization of combs studied in [6] where the base was $\mathbb{Z}^1$. We prove inequalities between generating functions for first return probabilities on bundled structures which allow us to investigate properties of the spectral dimension and to find bounds on the spectral dimension of random brushes.

In Chapter 3 we study non-generic trees which are a special type of simply generated trees. Simply generated trees are random trees of a fixed size where each tree is given a weight which depends only on the order of its vertices. Simply generated trees correspond to critical or subcritical Galton-Watson processes which are conditioned on the total progeny. In [7] generic trees are studied and it is shown that their probability measure, when the size goes to infinity, is concentrated on trees with exactly one infinite branch with finite critical Galton-Watson outgrowths. Their spectral dimension is shown to be $d_s = 4/3$. We will generalize this partially to certain kind of non-generic trees which still correspond to critical Galton-Watson processes. We then conjecture that there is a limiting measure on non-generic trees which correspond to subcritical Galton-Watson processes using arguments from [14–16]. The trees are characterized by having exactly one vertex of infinite order with subcritical Galton-Watson outgrowths. We find that their Hausdorff dimension is $d_H = \infty$ and that their spectral dimension obeys $d_s \geq 2$ with a model dependent upper bound.

2

Spectral dimension of random brushes

In [6] the spectral dimension of various ensembles of random combs was calculated. In this part we generalize the monotonicity results of [6] which allows us to find bounds on the spectral dimensions of graphs which we call brushes and define below.

A graph is a set of vertices linked together by a set of edges. In this section we only consider graphs which are locally finite, i.e. each vertex is linked to only finitely many other vertices, and connected meaning that any vertex can be reached from another vertex by following edges. For convenience we single out one vertex and call it the root. A simple random walk on a graph G starts at the root and travels to adjacent vertices with equal probability in discrete timesteps. Let $p_G(t)$ be the probability that a simple random walk on G is back at the root after t steps. If

$$p_G(t) \sim t^{-d_s/2} \tag{2.1}$$

as $t \to \infty$ then we say that d_s is the spectral dimension of the graph G. Here the meaning of $f(x) \sim x^\alpha$ as $x \to 0$ is that for any $\epsilon > 0$ there exist positive constants c_1 and c_2, which may depend on ϵ, such that

$$c_1 x^{\alpha+\epsilon} \leq f(x) \leq c_2 x^{\alpha-\epsilon} \tag{2.2}$$

for x small enough.

Some graphs have the property that every random walk beginning and ending at the root has an even number of steps. Then we have to replace $p_G(t)$ with $p_G(2t)$ in the above definition. In particular this is the case for brushes and trees.

The existence of d_s is not guaranteed for individual graphs but its ensemble average can be shown to be well defined in many cases [6, 7]. In the case of locally finite and connected graphs the spectral dimension is independent of the starting site of the random walk. To see this we let i be any vertex other than the root. We choose some path between r and i which has some length T. Let p_T and p'_T be the probabilities that a random walk follows this path from r to i and from i to r respectively. Let $p_{G,i}(t)$ be the probability that a random walk starting at i returns to i at time t. Then

$$p_G(t+2T) \geq p_T p_{G,i}(t) p'_T \geq p_T^2 p_G(t-2T)(p'_T)^2. \tag{2.3}$$

This shows that $p_{G,i}(t) \sim p_G(t)$ as $t \to \infty$.

Let us view $\mathbb{Z}^d$ as a graph with $j, k \in \mathbb{Z}^d$ neighbours if their distance is 1 and let the origin of $\mathbb{Z}^d$ be the root. The probability of a random walk returning to the root on $\mathbb{Z}^d$ after t steps has the property that

$$p_{\mathbb{Z}^d}(2t)t^{d/2} \to C(d) \tag{2.4}$$

as $t \to \infty$ where $C(d)$ only depends on d. Therefore the spectral dimension of $\mathbb{Z}^d$ is d.

Let N_l be a linear chain of length ℓ, i.e. the graph obtained be connecting nearest neighbours in $\{0, 1, \ldots, \ell\}$ with a link. Let 0 be the root of N_l. Similarly, let N_∞ be the infinite linear chain with root at 0. A d-brush is a graph constructed by attaching one of the graphs N_l, $l \in \mathbb{N}_0 \cup \{\infty\}$, to each vertex of $\mathbb{Z}^d$ by identifying the root of N_ℓ with a vertex in $\mathbb{Z}^d$, $l = 0$ corresponding to the empty chain. In a brush B we will refer to $\mathbb{Z}^d$ as the base and the linear chains as bristles.

A random brush is defined by letting the length of the bristles be identically and independently distributed by a probability measure on $\mathbb{N}_0 \cup \{\infty\}$. The case $d = 1$ corresponds to the combs studied in [6] which were shown to have a spectral dimension in the interval $[1, \frac{3}{2}]$. We will show that the spectral dimensions of random brushes satisfy

$$\begin{aligned}1\leq\ & d_s && \leq\frac{3}{2}, && \text{if} \quad d=1,\\ & d_s && =2, && \text{if} \quad d=2,\\ \frac{5}{2}\leq\ & d_s && \leq 3, && \text{if} \quad d=3,\\ 3\leq\ & d_s && \leq d, && \text{if} \quad d\geq 4. \end{aligned} \tag{2.5}$$

In the next section we define the generating functions we use to analyze the spectral dimension. We then establish a generalized monotonicity lemma which will directly imply the stated bounds on d_s.

2.1 Generating functions

Consider a locally finite and connected graph G. Let $p_G^1(t)$ be the probability that a random walk is at the root at time t the first time after $t=0$ ($p_G^1(0)=0$). We define the return generating function

$$Q_G(z)=\sum_{t=0}^{\infty} p_G(t)z^t \tag{2.6}$$

and the first return generating function

$$P_G(z)=\sum_{t=0}^{\infty} p_G^1(t)z^t. \tag{2.7}$$

By decomposing a return to the root into first return, second return etc. we find that the return generating function can be written as

$$Q_G(z)=\sum_{n=0}^{\infty} P_G(z)^n=\frac{1}{1-P_G(z)} \tag{2.8}$$

where the exponent n in the sum counts the contribution from the n-th return and the geometric sum is calculated in the second step.

The function $P_G(z)$ is analytic in the unit disc and $|P(z)|<1$ for $|z|<1$. If $P_G(z)\to 1$ as $z\to 1$ then $Q_G(z)$ clearly diverges in which case the random walk is recurrent and returns to the root eventually with probability one. If $P_G(z)\not\to 1$

as $z \to 1$ then the random walk is transient and returns to the root eventually with probability less than one. If G has a spectral dimension d_s then by integral comparison we see that

$$Q_G^{(n)}(z) \sim \begin{cases} 1 & \text{if } n = d_s/2 - 1 \\ (1-z)^{d_s/2-1-n} & \text{otherwise} \end{cases} \tag{2.9}$$

where n is the smallest nonnegative integer for which $Q_G^{(n)}(z)$ diverges as $z \to 1$.

When the generating functions are even functions of z it is convenient to introduce a new variable x through

$$z^2 = 1 - x \tag{2.10}$$

where $x \in [0,1]$. This is always the case for graphs which have the property that every random walk beginning and ending at the root has an even number of steps. We will denote the generating functions in x with the same symbol as the generating functions in z which hopefully causes no confusion. In the same way as above we see that if G has a spectral dimension d_s then

$$Q_G^{(n)}(x) \sim \begin{cases} 1 & \text{if } n = d_s/2 - 1 \\ (-1)^n x^{d_s/2-1-n} & \text{otherwise} \end{cases} \tag{2.11}$$

where n is the smallest nonnegative integer for which $Q_G^{(n)}(x)$ diverges as $x \to 0$.

In some cases it is possible to find a nice formula for the generating functions. Take for example the linear graph N_l. By decomposing a first return random walk on N_l into a first step, then arbitrary many first returns to the next neighbour of the root and finally a last step back to the root we get the following recurrence relation for the first return probability generating functions of N_l

$$P_{l+1}(x) = \frac{1-x}{2-P_l(x)}, \qquad l \geq 1. \tag{2.12}$$

with boundary condition $P_l(x) = 1 - x$. This is solved in [6] for finite and infinite l giving

$$P_l(x) = 1 - \sqrt{x}\frac{(1+\sqrt{x})^l - (1-\sqrt{x})^l}{(1+\sqrt{x})^l + (1-\sqrt{x})^l} \tag{2.13}$$

and

$$P_\infty(x) = 1 - \sqrt{x}. \tag{2.14}$$

respectively. This shows that the graphs are recurrent and for every finite l the spectral dimension is zero but for the infinite half line the spectral dimension is one.

2.1.1 Random brushes

Let μ be a probability measure on $\mathbb{N}_0 \cup \{\infty\}$. Let $\mathcal{B}^d$ be the set of all d-brushes. We define a probability measure π on $\mathcal{B}^d$ by letting the measure of the set of d-brushes Ω which have bristles at $n_1, n_2, ..., n_k \in \mathbb{Z}^d$ of length $\ell_1, \ell_2, ..., \ell_k$ be

$$\pi(\Omega) = \prod_{i=1}^{k} \mu(l_i). \tag{2.15}$$

This formula defines the measure π uniquely. The set $\mathcal{B}^d$ together with π is a random brush. We define the averaged generating functions

$$\bar{P}(x) = \langle P_B(x) \rangle_\pi \tag{2.16}$$

and

$$\bar{Q}(x) = \langle Q_B(x) \rangle_\pi \tag{2.17}$$

where $\langle \cdot \rangle_\pi$ denotes expectation with respect to π. We say that a random brush has the spectral dimension d_s if $\bar{Q}(x)$ obeys the relation (2.11).

2.2 A generalized monotonicity lemma

In [6] it was shown that the first return generating function $P(x)$ is a decreasing function of the length of the teeth attached to the base. A similar result was obtained in [7] for trees showing that $P(x)$ decreases when branches are added to a tree. In this section we prove similar results for more general graphs. Lemma 1 deals with recurrent bases and Lemma 2 deals with transient bases.

Let G_1 and G_2 be rooted graphs. Assume that G_1 can be constructed from G_2 by attaching rooted graphs $F(i)$ by their roots to sites $i \neq r$ of G_2. We call the graph G_1 a bundled structure with base G_2 and fibers $F(i)$. Let the roots of G_1 and G_2 be the same vertex (regarding G_2 as a subgraph of G_1).

Lemma 1

$$P_{G_1}(z) \leq P_{G_2}(z) \tag{2.18}$$

with equality if and only if all the $F(i)$'s are recurrent and $z = 1$.

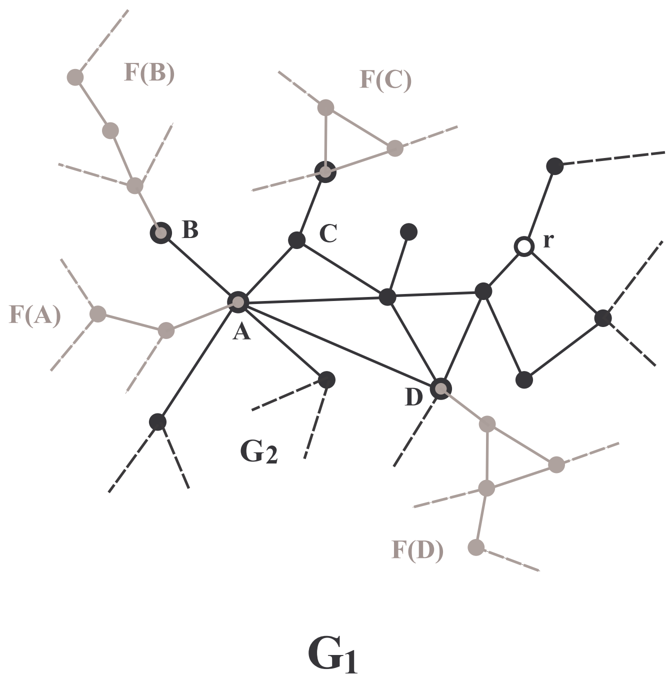

Figure 2.1: An example of a bundled structure G_1 constructed from G_2 and the $F(i)$'s.

Proof: We can write $P_{G_2}(z)$ as the sum over random walks ω which start and end at the root without intermediate visits to the root. This condition is denoted 'ω: FR on G_2' where FR stands for 'first return'. Each walk has a weight which is the product of one over the order of vertices visited by the walk

$$W_{G_2}(\omega) = \prod_{t=0}^{|\omega|-1} (\sigma_{G_2}(\omega_t))^{-1} \tag{2.19}$$

and each step of a walk has a factor z associated with it so

$$P_{G_2}(z) = \sum_{\omega:\ \mathrm{FR\ on}\ G_2} W_{G_2}(\omega) z^{|\omega|} \tag{2.20}$$

where $\sigma_{G_2}(\omega_t)$ is the order of the vertex ω_t on G_2 where the walk ω is located at time t and $|\omega|$ is the number of steps in ω.

Now consider a random walk ω' on G_1 which starts at the root. Let ω be the subwalk of ω' which only travels on G_2. If we look at the walk ω at time t and location ω_t then ω can be a subwalk of many different walks ω' corresponding to all possible visits into the graph $F(\omega_t)$ before returning back to the walk on G_2. The weight of these visits is

$$\sum_{n=0}^{\infty}\Big(\frac{\sigma_{F(\omega_t)}(\omega_t)}{\sigma_{G_1}(\omega_t)}P_{F(\omega_t)}(z)\Big)^n = \frac{1}{1-\Big(\frac{\sigma_{F(\omega_t)}(\omega_t)}{\sigma_{G_1}(\omega_t)}P_{F(\omega_t)}(z)\Big)} \tag{2.21}$$

where n counts the number of visits and the factor in front of $P_{F(\omega_t)}(z)$ changes the order of the root of $F(\omega_t)$ to $\sigma_{G_1}(\omega_t) = \sigma_{G_2}(\omega_t) + \sigma_{F(\omega_t)}(\omega_t)$. The weight of the first step back into G_2 after these visits to $F(\omega_t)$ is

$$\frac{1}{\sigma_{G_1}(\omega_t)}z. \tag{2.22}$$

Now replace the original weight $\sigma_{G_2}(\omega_t)^{-1}z$ of ω at each point $\omega_t \neq \omega_0$ by the product of the factors (2.21) and (2.22). This newly weighted ω then accounts for every random walk on G_1 which has ω as a subwalk on G_2. Thus we can write

$$\begin{aligned} P_{G_1}(z) &= \sum_{\omega:\text{ FR on } G_2} \sigma_{G_2}(\omega_0)^{-1}z\prod_{t=1}^{|\omega|-1}\Big(\frac{z}{\sigma_{G_2}(\omega_t)+\sigma_{F(\omega_t)}(\omega_t)(1-P_{F(\omega_t)}(z))}\Big) \\ &= \sum_{\omega:\text{ FR on } G_2} K_{G_1,G_2}(z;\omega)W_{G_2}(\omega)z^{|\omega|} \end{aligned} \tag{2.23}$$

where in the last step we defined

$$K_{G_1,G_2}(z;\omega) = \prod_{t=1}^{|\omega|-1}\Big(\frac{\sigma_{G_2}(\omega_t)}{\sigma_{G_2}(\omega_t)+\sigma_{F(\omega_t)}(\omega_t)(1-P_{F(\omega_t)}(z))}\Big). \tag{2.24}$$

Since $P_{F(\omega_t)}(z) \leq 1$ with equality if and only if $F(\omega_t)$ is recurrent and $z = 1$ it is clear that $K_{G_1,G_2}(z;\omega) \leq 1$ for all z with equality if and only if all the graphs $F(\omega_t)$ for a given ω on G_2 are recurrent and $z = 1$. When we consider all such random walks we get the inequality (2.18).

□

Lemma 2 *If there exists an $n \geq 1$ such that $P_{G_2}^{(n-1)}(z)$ is continuous on the closed interval $[0,1]$ and if all the $F(i)$'s are recurrent then for any $z \in]0,1[$ there exists a $\xi \in]z,1[$ such that*

$$P_{G_1}^{(n)}(\xi) \geq P_{G_2}^{(n)}(\xi). \tag{2.25}$$

Proof: We define

$$H_{G_1,G_2}(z;n) = \sum_{\omega:\ \mathrm{FR\ on}\ G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \frac{d^{n-1}}{dz^{n-1}} z^{|\omega|} \tag{2.26}$$

where K_{G_1,G_2} is defined as above. Every derivative of a (first) return generating function is a positive increasing function of $z \in [0,1[$ since the power series have no negative coefficients. It is easy to verify that the function $K_{G_1,G_2}(z)$ has the same properties. Therefore we get by differentiating (2.23) n times

$$\begin{aligned}
P_{G_1}^{(n)}(z) &= \sum_{i=0}^{n} \binom{n}{i} \sum_{\omega:\ \mathrm{FR\ on}\ G_2} K_{G_1,G_2}^{(i)}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n-i)} \\
&\geq \sum_{\omega:\ \mathrm{FR\ on}\ G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n)} \\
&+ \quad n \sum_{\omega:\ \mathrm{FR\ on}\ G_2} K'_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n-1)} \\
&\geq \sum_{\omega:\ \mathrm{FR\ on}\ G_2} K_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n)} \\
&+ \sum_{\omega:\ \mathrm{FR\ on}\ G_2} K'_{G_1,G_2}(z;\omega) W_{G_2}(\omega) \left(z^{|\omega|}\right)^{(n-1)} \\
&= H'_{G_1,G_2}(z;n).
\end{aligned} \tag{2.27}$$

In the first step we used the binomial formula for the n-th derivative of a product. In the second step every term of the binomial sum was thrown away except for $i = n$ and $i = n-1$. In the third step the n in front of the second sum was replaced by one and the final step is obvious from the definition of $H_{G_1,G_2}(z;n)$.

With the same argument as in the proof of Lemma 1 it holds that $H_{G_1,G_2}(z;n) \leq P_{G_2}^{(n-1)}(z)$. We have equality when $z = 1$ since all the $F(i)$'s are recurrent and because $P_{G_1}^{(n-1)}(z)$ and therefore also $H_{G_1,G_2}(z;n)$ are continuous on $[0,1]$. Then since $H_{G_1,G_2}(z;n)$ and $P_{G_2}^{(n-1)}(z)$ are positive and increasing functions of z we get that

$$\frac{H_{G_1,G_2}(1;n) - H_{G_1,G_2}(z;n)}{P_{G_2}^{(n-1)}(1) - P_{G_2}^{(n-1)}(z)} \geq 1. \tag{2.28}$$

By a generalized mean-value theorem [17] there exists a $\xi \in]z, 1[$ such that

$$\frac{H_{G_1,G_2}(1;n) - H_{G_1,G_2}(z;n)}{P_{G_2}^{(n-1)}(1) - P_{G_2}^{(n-1)}(z)} = \frac{H'_{G_1,G_2}(\xi;n)}{P_{G_2}^{(n)}(\xi)}. \tag{2.29}$$

Then for any $z \in]0, 1[$ there exists a $\xi \in]z, 1[$ such that

$$P_{G_2}^{(n)}(\xi) \leq H'_{G_1,G_2}(\xi;n) \leq P_{G_1}^{(n)}(\xi). \tag{2.30}$$

□

From the above lemmas we get the following theorem.

Theorem 1 *Assume that all the $F(i)$'s are recurrent and that G_1 and G_2 have spectral dimensions d_{s_1} and d_{s_2} respectively. If G_2 is recurrent then G_1 is recurrent and $d_{s_1} \geq d_{s_2}$. If G_2 is transient then G_1 is transient and $d_{s_1} \leq d_{s_2}$.*

Proof: First consider the case when G_2 is recurrent. If all the $F(i)$'s are recurrent Lemma 1 shows that $P_{G_1}(1) = P_{G_2}(1) = 1$ and therefore G_1 is also recurrent. Assuming the existence of d_{s_1} and d_{s_2} and using (2.9) and Lemma 1 along with (2.8) we get

$$c_1(1-z)^{d_{s_1}/2-1+\epsilon} \leq Q_{G_1}(z) \leq Q_{G_2}(z) \leq c_2(1-z)^{d_{s_2}/2-1-\epsilon} \tag{2.31}$$

for z close to 1 where $\epsilon > 0$ is arbitrary and c_1 and c_2 are positive constants which may depend on ϵ. Then

$$(1-z)^{\frac{1}{4}(d_{s_2}-d_{s_1})-\epsilon} > c \tag{2.32}$$

where c is a positive constant. By choosing $\epsilon < \frac{1}{4}|d_{s_2} - d_{s_1}|$ and sending $z \to 1$ we see that it must hold that $d_{s_1} \geq d_{s_2}$.

Now consider the case when G_2 is transient. Again, if all the $F(i)$'s are recurrent Lemma 1 shows that $P_{G_1}(1) = P_{G_2}(1) < 1$ and therefore G_1 is also transient. First note that if some n-th derivative $Q_{G_i}^{(n)}(z),\ i = 1, 2$ diverges as $z \to 1$ then from (2.9) we get

$$Q_{G_i}^{(n)}(z) \sim \frac{P_{G_i}^{(n)}(z)}{(1-P_{G_i}(z))^2} \sim P_{G_i}^{(n)}(z) \qquad \text{as} \quad z \to 1 \tag{2.33}$$

since $P_{G_i}(1) < 1$. By Lemma 2 there exists a sequence $\xi_k < 1$ such that $\xi_k \to 1$ as $k \to \infty$ and

$$P_{G_2}^{(n)}(\xi_k) \leq P_{G_1}^{(n)}(\xi_k) \tag{2.34}$$

for all k where n is the lowest positive integer for which $P_{G_2}^{(n)}(z)$ diverges as $z \to 0$. We then see that $P_{G_1}^{(n)}(z)$ also diverges as $z \to 1$ and if n is not the lowest integer for which that happens then clearly $d_{s_1} < d_{s_2}$. If however n is also the lowest integer for which $P_{G_1}^{(n)}(z)$ diverges then we get from (2.34), (2.33) and (2.9) that

$$c_1(1-\xi_k)^{d_{s_2}/2-1-n+\epsilon} \leq c_2(1-\xi_k)^{d_{s_1}/2-1-n-\epsilon} \tag{2.35}$$

for k large enough where $\epsilon > 0$ is arbitrary and c_1 and c_2 are positive constants which may depend on ϵ. With the same arguments as before we choose $\epsilon < \frac{1}{4}|d_{s_2} - d_{s_1}|$ and let $k \to \infty$ to see that $d_{s_1} \leq d_{s_2}$.

$\square$

It is not surprising that attaching recurrent fibers to a recurrent base results in a recurrent graph. If a random walker on the base happens to travel into a fibre he will eventually return back to the base with probability one. However the meeting with the fiber delays the walker and therefore increases the spectral dimension. In the case of a transient base the time spent in the recurrent fibre reduces the time spent in the base and therefore the probability of not returning to the root. Therefore the spectral dimension decreases.

2.2.1 Monotonicity results for random brushes

Now, let's consider the case when $G_2 = \mathbb{Z}^d$ and instead of having a fixed G_1 we consider a random d-brush $(\mathcal{B}^d, \pi)$. We would like to get similar results for random brushes as in Lemmas 1 and 2. First we note that by Lemma 1 we have for any $B \in \mathcal{B}^d$ that

$$P_{*d}(z) \leq P_B(z) \leq P_{\mathbb{Z}^d}(z) \tag{2.36}$$

where $*d$ is the full brush, defined in Section 2.3.1. By integrating with respect to π we get

$$P_{*d}(z) \leq \overline{P}(z) \leq P_{\mathbb{Z}^d}(z). \tag{2.37}$$

To get a similar result for random d-brushes as in Lemma 2 we consider the case $d > 2$ and we define the functions

$$\overline{H}_a(z;n) = \int_\pi H_{B,\mathbb{Z}^d}(z;n)d\pi(B) \qquad \text{and} \qquad \overline{H}_b(z) = \int_\pi H_{*d,B}(z;1)d\pi(B) \quad (2.38)$$

where the function in the integrand is defined as in (2.26) and n is the smallest positive integer for which $P^{(n)}_{\mathbb{Z}^d}(z)$ diverges as $z \to 1$. With the same calculation as in (2.27) we get

$$\frac{\overline{H}'_a(z,n)}{\overline{P}^{(n)}(z)} \leq 1 \qquad \text{and} \qquad \frac{\overline{H}'_b(z)}{P'_{*d}(z)} \leq 1. \quad (2.39)$$

We clearly have $\overline{H}_a(z;n) \leq P^{(n-1)}_{\mathbb{Z}^d}(z)$ and $\overline{H}_b(z) \leq \overline{P}(z)$ both with equality when $z = 1$ because the bristles are recurrent. Since the functions $\overline{H}_a(z;n)$,$P^{(n-1)}_{\mathbb{Z}^d}(z)$, $\overline{H}_b(z)$ and $\overline{P}(z)$ are all increasing functions of z on $[0,1[$ we get with the same argument as in (2.29) that for any $z \in]0,1[$ there exists a $\xi \in]z,1[$ such that

$$1 \leq \frac{\overline{P}^{(n)}(\xi)}{P^{(n)}_{\mathbb{Z}^d}(\xi)} \qquad \text{and} \qquad 1 \leq \frac{P'_{*d}(\xi)}{\overline{P}'(\xi)}. \quad (2.40)$$

These arguments can be generalized by replacing the base $\mathbb{Z}^d$ with any fixed graph and by replacing the random bristles by any random graph which consists of a probability distribution on a set of recurrent graphs.

2.3 Bounds on the spectral dimension

Now that we have established these monotonicity results we can find bounds on the spectral dimension of (random) brushes. First we find the spectral dimension of brushes which have every bristle infinite. We call such brushes full brushes and denote them $*d$. Then we use the monotonicity results to sandwich any brush between an empty brush and a full brush.

For those graphs which have the property that the (first) return generating function is an even function of z it is easy to verify that all the inequalities derived in the previous section hold for generating functions in the variable x defined in (2.10). This is the case for fixed and random brushes. For convenience we present the following calculations in the variable x.

2.3.1 The full brush

We can relate the first return generating function of the full d-brush to the first return generating functions of $\mathbb{Z}^d$ and N_∞. We use the same argument as in the proof of the monotonicity lemma. We simply replace all the graphs $F(i)$ with N_∞ and note that the order of every point in $\mathbb{Z}^d$ is $2d$. Then equation (2.23) becomes

$$\begin{aligned} P_{*d}(x) &= \left(1+\frac{1-P_\infty(x)}{2d}\right)\sum_{\omega:\ \text{FR on } \mathbb{Z}^d}\prod_{t=0}^{|\omega|-1}\frac{1}{2d}\frac{\sqrt{1-x}}{1+\frac{1-P_\infty(x)}{2d}} \\ &= \left(1+\frac{\sqrt{x}}{2d}\right)P_{\mathbb{Z}^d}(x_{\text{ren}}(x)) \end{aligned} \tag{2.41}$$

where we used (2.14) and defined x_{ren} through

$$\sqrt{1-x_{\text{ren}}} = \frac{\sqrt{1-x}}{1+\frac{\sqrt{x}}{2d}}. \tag{2.42}$$

We see that $x_{\text{ren}} = \sqrt{x}/d + O(x)$. By differentiating (2.41) once and comparing with (2.11) we find the spectral dimension of the full brush

$$d_* = \begin{cases} \frac{d}{2}+1 & \text{if } 1 \le d \le 4 \\ 3 & \text{if } d \ge 4. \end{cases} \tag{2.43}$$

Note that the spectral dimension is always $d_s = 3$ when $d \ge 4$. This comes from the fact that $x'_{\text{ren}}(x) \sim x^{-1/2}$ as $x \to 0$ and that $|P'_{\mathbb{Z}^d}(x)|$ grows at most like $-\ln(x)$ as $x \to 0$ when $d \ge 4$. If we replace the infinite bristles with finite ones, all of which have the same length, then with the same calculation we see that the spectral dimension remains equal to d.

These results are special cases of a more general result obtained in [8] for bundled structures. There, the base $\mathbb{Z}^d$ can be replaced by any connected graph G and the infinite bristle (fibre) can also be replaced by any fixed, connected graph F.

2.3.2 Results

Any fixed d-brush B can be constructed from $\mathbb{Z}^d$ by attaching (recurrent) bristles to it and the full d-brush can be constructed from B by attaching (recurrent) bristles to it. Therefore, by Theorem 1 in Section 2.2, the spectral dimension of any fixed d-brush, if it exists, lies between d and d_*. This also holds for random brushes as is clear from equations (2.37) and (2.40). The spectral dimension for any fixed or random d-brush, if it exists, therefore obeys the relation (2.5).

It is interesting to note that the spectral dimension of random 2-brushes always equals 2. In fact, from the relation (2.4) it follows from (2.42) and Lemma 1 that there exist positive constants c_1 and c_2 such that

$$c_1|\ln(x)| \leq \overline{P}(x) \leq c_2|\ln(x)| \tag{2.44}$$

for a random 2-brush when x is small enough . This is a more strict condition on the asymptotic behavior of $\overline{P}(x)$ than the condition that $\overline{P}(x) \sim 1$ as $x \to 0$. The reason why the spectral dimension is always 2 is that when we construct a 2-brush B by attaching bristles to $\mathbb{Z}^2$ we get a similar scenario as in (2.41) namely that $P_B(x) \sim P_{\mathbb{Z}^2}(x_{\text{ren}}(x))$ as $x \to 0$ where $x_{\text{ren}}(x)$ is some function of x. If $x_{\text{ren}}(x) \sim x^\alpha$ as $x \to 0$ then the logarithm in $P_{\mathbb{Z}^2}$ does not see the exponent α and behaves as if no bristles were attached.

It is also interesting that for $d \geq 4$ the lower bound on the spectral dimension always equals 3. In fact it is easy to see that attaching a single infinite bristle to $\mathbb{Z}^d$ with $d \geq 4$ reduces the spectral dimension to 3. We can show this by attaching the infinite bristle to the root of $\mathbb{Z}^d$ since the spectral dimension is independent of the starting site of the random walks. We call the resulting graph $d_\perp$. The first return generating function for this graph is

$$P_{\perp d}(x) = \frac{2d-1}{2d} P_{\mathbb{Z}^d}(x) + \frac{1}{2d} P_\infty(x). \tag{2.45}$$

Since $d \geq 4$ equations (2.11) and (2.4) show that $|Q'_{\mathbb{Z}^d}(x)|$ diverges at most as $-\ln(x)$ as $x \to 0$ which is slower than the divergence of $Q'_\infty(x)$. Therefore by differentiating (2.45) we get

$$Q'_{\perp d}(x) \sim P'_{\perp d}(x) \sim P'_\infty(x) \sim x^{-1/2} \tag{2.46}$$

as $x \to 0$ and therefore by (2.11) the spectral dimension is $d_\perp = 3$. From this and the lower bound in (2.5) it follows from equations similar to (2.40) that if a random d-brush with $d \geq 4$ has a nonzero probability of having one or more infinite bristles its spectral dimension equals 3.

We find with similar arguments that adding a single (or finitely many) bristles to $\mathbb{Z}^3$ gives the spectral dimension 3. However if we add infinitely many bristles the spectral dimension can be lowered as is seen e.g. in the case of the full 3-brush.

3

Non-generic trees

A tree is a connected graph with no loops. We consider rooted planar trees where the root has order one and is denoted r. Planar means that if we imagine the trees to be embedded in the plane then two trees are the same if one can be deformed into the other without links crossing each other. Let Γ_N be the set of all such trees having N links and let Γ be the set of all locally finite rooted planar trees. We define a metric on Γ by

$$d_\Gamma(\tau, \tau') = \inf_{R \geq 0} \{ \frac{1}{R+1} | B_R(\tau) = B_R(\tau') \} \tag{3.1}$$

where $B_R(\tau)$ is the subtree of τ spanned by vertices at distance less than or equal to R from the root. We denote the number of links in a tree τ with $|\tau|$ and refer to it as the size of the tree.

In this section we study a model of random trees which are often called simply generated trees. It is defined by a set of positive branching weights $w_n, n \geq 1$. Given these branching weights we define the finite volume partition function for trees of size N

$$Z_N = \sum_{\tau \in \Gamma_N} \prod_{i \in \tau \setminus r} w_{\sigma_i} \tag{3.2}$$

and a probability distribution ν_N on Γ_N by

$$\nu_N(\tau) = Z_N^{-1} \prod_{i \in \tau \setminus r} w_{\sigma_i}, \qquad \tau \in \Gamma_N. \tag{3.3}$$

The set Γ_N equipped with the probability measure ν_N is our model of a random tree of size N. We are interested in determining how a typical tree looks like when N

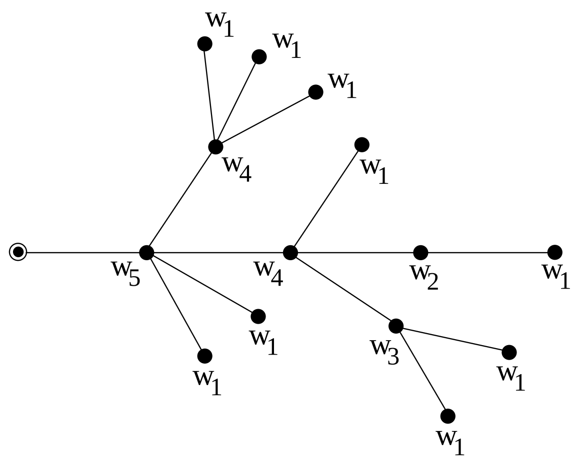

Figure 3.1: A tree with weight $w_1^9\ w_2\ w_3\ w_4^2\ w_5$.

is large or even when $N \to \infty$. Some desireable parameters would be the Hausdorff and spectral dimension. In [18] the case $w_n = 1$, $\forall n$ is studied. There it is shown that when $N \to \infty$ the probability measure ν_N converges weakly to a probability measure on Γ which is concentrated on trees with one infinite branch with finite outgrowths. In [7] the same is shown to be true for so called generic trees which are defined in the next section. There it is established that the spectral dimension is $d_s = 4/3$. The Hausdorff dimension of generic trees is $d_H = 2$ [2]. In this part of the thesis we discuss what is known about non-generic trees and try to find similar results as in the generic case.

3.1 Some useful tools

We define a generating function for the branching weights

$$g(z) = \sum_{n=1}^{\infty} w_n z^{n-1} \tag{3.4}$$

and a generating function for the finite volume partition function

$$Z(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N. \tag{3.5}$$

The contribution to Z from trees for which the vertex next to the root has order k is $\zeta w_k Z(\zeta)^{k-1}$. By summing over k we get the following relation between the generating functions

$$Z(\zeta) = \zeta \sum_{k=1}^{\infty} w_k Z(\zeta)^{k-1} = \zeta g(Z(\zeta)). \tag{3.6}$$

Let ρ and ζ_0 be the radii of convergence of the generating functions g and Z respectively. Here we will always consider branching weights such that $\rho > 0$. We define $Z_0 = \lim_{\zeta \to \zeta_0} Z(\zeta)$. From the above relation we see that Z_0 is finite and

$$Z_0 \leq \rho. \tag{3.7}$$

When $Z_0 < \rho$ we have a generic ensemble of infinite trees but when the equality holds we have a non-generic ensemble. The generic case is easier to analyze because the function g is analytic in a neighbourhood of a disk centered at zero and with radius Z_0. Note that when ρ is infinite we always have a generic ensemble.

From the functional equation (3.6) we can relate the coefficients of powers of $Z(\zeta)$ to the branching weights w_n. By Lagrange's Inversion Theorem (see e.g. [19]) we get

$$[\zeta^N]\{Z(\zeta)^k\} = \frac{k}{N} \sum_{N_1+\ldots+N_N=N-k} \prod_{i=1}^{N} w_{N_i+1} = \frac{k}{N}[z^{N-k}]\{g(z)^N\} \tag{3.8}$$

where $[z^n]\{f(z)\}$ stands for the n-th coefficient of the power series $f(z)$. The case $k=1$ gives Z_N.

3.2 Galton-Watson processes

In this section we discuss a relation between simply generated trees and so called Galton-Watson processes which can give us some insight in how simply generated trees look like in the large N limit. A Galton-Watson process is a process for tree growth which was first studied by Galton and Watson in the late 19th century in relation to family trees. Since then this process has for example been a model for populations of neutrons, genes, cosmic rays and more. Standard references for Galton-Watson processes are e.g. [20, 21].

The process is defined in the following way. We start with a single ancestor (in general they can be many) which has n offsprings with probability p_n where p_n are non-negative numbers and

$$\sum_{n=0}^{\infty} p_n = 1. \tag{3.9}$$

Each offspring then has n offsprings itself with the same probabilities p_n and so on. For convenience we add a root r to the Galton-Watson trees by linking a vertex of order one to the ancestor. The process gives a probability measure on the set of all finite trees

$$\mu(\tau) = \prod_{i \in \tau \setminus r} p_{\sigma_i - 1}, \qquad \text{where} \qquad \tau \in \bigcup_{N=0}^{\infty} \Gamma_N. \tag{3.10}$$

We define a generating function for the offspring probabilities

$$f(z) = \sum_{n=0}^{\infty} p_n z^n. \tag{3.11}$$

Galton-Watson processes are usually divided into three categories depending on the size of the first moment of the generating function $m = f'(1)$. It is clear that m represents the mean number of offsprings of each individual. If $m > 1$ the process is said to be supercritical and the probability that it survives forever is positive. If $m = 1$ the process is said to be critical and it dies out eventually with probability one. If $m < 1$ the process is said to be subcritical and it dies out eventually with probability one and much faster than in the critical case.

The reason why we are interested in Galton-Watson processes in this paper is the following relation.

Lemma 3 *A simply generated tree of size N is a rooted Galton-Watson process with offspring probabilities*

$$p_n = \zeta_0 w_{n+1} Z_0^{n-1} \tag{3.12}$$

which is conditioned on the total size of the trees. The Galton-Watson process can be either critical or subcritical.

Proof: With the p_n given in (3.12) we get

$$\sum_{n=0}^{\infty} p_n = \zeta_0 \sum_{n=0}^{\infty} w_{n+1} Z_0^{n-1} = \zeta_0 Z_0^{-1} g(Z_0) = 1 \tag{3.13}$$

by using (3.6). Therefore the p_n are Galton-Watson offspring probabilities. The first moment is

$$m = \sum_{n=0}^{\infty} n p_n = \zeta_0 \sum_{n=0}^{\infty} n w_{n+1} Z_0^{n-1} = \zeta_0 g'(Z_0) = Z_0 \frac{g'(Z_0)}{g(Z_0)}. \tag{3.14}$$

By differentiating (3.6) with respect to ζ and rearranging terms we find that

$$Z(\zeta)\frac{g'(Z(\zeta))}{g(Z(\zeta))} = 1 - \frac{g(Z(\zeta))}{Z'(\zeta)} \leq 1 \tag{3.15}$$

and the equality holds for $\zeta = \zeta_0$ if and only if $Z'(\zeta_0) = \infty$. This shows that the process is critical if $Z'(\zeta_0) = \infty$ and subcritical otherwise.

The measure corresponding to these probabilities when conditioned on trees of size N is then

$$\mu_N(\tau) = C_N \prod_{i\in\tau\setminus r} p_{\sigma_i - 1} = C_N Z_0^{-1}\zeta_0^N \prod_{i\in\tau\setminus r} w_{\sigma(i)} \tag{3.16}$$

where $\tau \in \Gamma_N$ and C_N is a normalization constant. From (3.3) we see that $\nu_N = \mu_N$ and $C_N = Z_0\zeta_0^{-N} Z_N^{-1}$ which proves the lemma.

□

Since critical and subcritical Galton-Watson processes are relevant when dealing with simply generated trees we state here some results about standard properties proved e.g. in [20]. Let $\langle\cdot\rangle_\mu$ denote expectation with respect to the measure μ defined in (3.10). Let $h(\tau)$ denote the maximum graph distance from the root to any vertex of τ, referred to as the height of τ.

Lemma 4 *For subcritical Galton-Watson trees with mean number of offsprings m it holds that*

$$\langle |B_R| \rangle_\mu = \frac{1-m^{R+1}}{1-m} \tag{3.17}$$

and letting $R \to \infty$ we find that the expectation value of the size of trees is finite

$$\langle |B_\infty| \rangle_\mu = \frac{1}{1-m}. \tag{3.18}$$

Lemma 5 *For critical Galton-Watson trees it holds that*

$$\langle |B_R| \rangle_\mu = R \tag{3.19}$$

and if $f''(1) < \infty$

$$\mu(\{\tau \in \Gamma | h(T) > R\}) = \frac{2}{f''(1)R} + O(R^{-2}). \tag{3.20}$$

The case when $f''(1)$ is infinite in critical processes has been studied e.g. in [22]. There, generating functions of the form

$$f(s) = s + (1-s)^{1+\alpha} L(1-s) \tag{3.21}$$

are studied where $0 < \alpha \leq 1$ and L is slowly varying. Slowly varying means that

$$\frac{L(\lambda t)}{L(t)} \to 1 \qquad \text{as} \qquad t \to \infty \tag{3.22}$$

for all $\lambda > 0$. It is shown that

$$\mu(\{\tau \in \Gamma | h(T) > R\})^\alpha L(\mu(\{\tau \in \Gamma | h(T) > R\})) \approx \frac{1}{\alpha R} \tag{3.23}$$

as $R \to \infty$. The generic behaviour in (3.20) therefore changes and becomes model dependent. Here the meaning of $f(x) \approx g(x)$ as $x \to \infty$ is that

$$\frac{f(x)}{g(x)} \to 1 \qquad \text{as} \quad x \to \infty. \tag{3.24}$$

3.3 The generic case

Generic random trees are defined by the condition $Z_0 < \rho$ as was explained above. In this case it can be shown [23] that Z_N has the generic behaviour

$$Z_N = C N^{-3/2} \zeta_0^{-N} (1 + O(N^{-1})) \tag{3.25}$$

where C is a constant independent of N. This immediately shows that $Z'(\zeta) \to \infty$ as $\zeta \to \zeta_0$ and thus generic trees correspond to a critical Galton-Watson process conditioned on the total size N.

It is shown in [7] using the methods of [18] that the probability measure ν_N for generic trees converges weakly to a probability measure ν on Γ. This means that

$$\int_\Gamma f d\nu_N \to \int_\Gamma f d\nu, \qquad \text{as} \qquad N \to \infty \tag{3.26}$$

for all bounded functions f on Γ which are continuous in the topology defined by the metric d_Γ. Furthermore the measure is shown to be concentrated on trees with one infinite branch growing from the root with identical and independent critical Galton-Watson outgrowths distributed by (3.10).

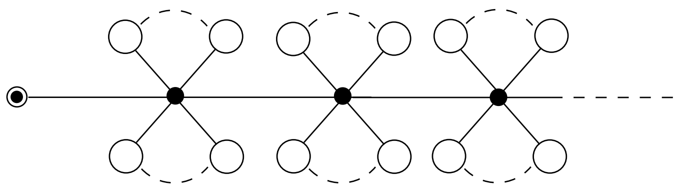

Figure 3.2: A generic tree consists of one infinite branch with critical Galton-Watson outgrowths. The balloons denote Galton-Watson trees.

The probability of having k left branches and l right branches growing from a vertex on the infinite branch is

$$\phi(k,l) = \zeta_0 w_{2+k+l} Z_0^{k+l}. \tag{3.27}$$

The outgrowths are free in the sense that there is no condition on their size. We can understand this in the following way. As N goes to infinity the size constraint on the Galton-Watson process is completely taken care of by the one infinite branch. The rest of the graph then grows freely like a critical Galton-Watson process.

As is explained in [18], to prove the convergence of the measure it is sufficient to show that for any value of $R \geq 0$ the following holds

Property 1

$$\nu_N(\{\tau \in \Gamma : |B_R(\tau)| > K\}) \to 0 \qquad \textit{as} \qquad K \to \infty \tag{3.28}$$

uniformly in N.

Property 2 *The sequence*

$$(\nu_N(\{\tau \in \Gamma : B_R(\tau) = \tau_0\}))_{N\in\mathbb{N}} \tag{3.29}$$

is convergent for each finite tree $\tau_0 \in \Gamma$.

Both properties are proven for generic trees in Appendix A in [7]. The first property shows that the order of vertices stays finite as $N \to \infty$. This seems to fail in some cases for non-generic trees as will be discussed later. The second property is also true in many non-generic cases as will now be proved.

Assume that Z_N has the asymptotic behaviour

$$Z_N \approx CN^{-\delta}\zeta_0^{-N}L(N) \tag{3.30}$$

for large N, where $C > 0$ is a constant and L is slowly varying. We also assume that

$$\max_{aN\leq N'\leq N}\left(\frac{L(N')}{L(N)}\right) < D, \qquad \text{for all} \quad N \tag{3.31}$$

where $D > 0$ and $0 < a < 1$ are constants and that $L(N)$ grows or decays slower than any power of N. This is for example true for any power of logarithms. In the generic case we always have $\delta = 3/2$ but in non-generic ensembles the existence of δ is not always guaranteed. However it seems to be possible to construct non-generic models with any $\delta \geq 3/2$ as we shall later see.

Let τ_0 be a finite graph and let M be the number of vertices in τ_0 at graph distance R from the root. We can decompose any tree τ for which $B_R(\tau) = \tau_0$ into the tree τ_0 and rooted subtrees whose roots are at graph distance $R-1$ from the root of τ_0 (see Figure 3.3). Note that the roots of these subtrees are vertices of τ_0. Then we can write

$$\nu_N(\{\tau \in \Gamma : B_R(\tau) = \tau_0\}) = W(\tau_0)Z_N^{-1}\sum_{N_1+\ldots+N_M=N+M-|\tau_0|}\prod_{i=1}^{M} Z_{N_i} \tag{3.32}$$

where

$$W(\tau_0) = \prod_{i\in B_{R-1}(\tau_0)\setminus r} w_{\sigma(i)} \tag{3.33}$$

is the contribution from vertices in τ_0 at a distance less than R from the root. The Z_{N_i} in the last product in (3.32) is the contribution from the subtree attached to

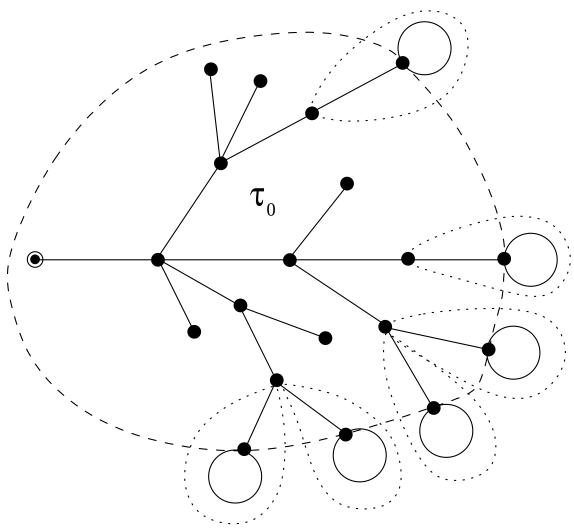

Figure 3.3: The tree τ_0 in the case $R = 4$ and $M = 6$. The balloons denote all possible rooted trees attached to τ_0 at a distance $R = 4$ and their roots are in τ_0.

vertex i of τ_0.

Now choose a positive constant A. The contribution to (3.32) from terms for which $N_i \geq (N + M - \tau_0)/M$ and $N_j \geq A$ for some pair of indices $i \neq j$ can be estimated from above with

$$
\begin{aligned}
& W(\tau_0)M^2 \sum_{\substack{N_1+\ldots+N_M=N+M-|\tau_0| \\ N_1\geq(N+M-\tau_0)/M, N_2\geq A}} Z_N^{-1} \prod_{i=1}^{M} Z_{N_i} \\
\leq\ & W(\tau_0)M^2\zeta_0^{M-|\tau_0|} \left(\frac{NM}{N+M-|\tau_0|}\right)^{\delta} \max_{N_1} \left(\frac{L(N_1)}{L(N)}\right) \sum_{\substack{N_3,\ldots,N_M\geq 1 \\ N_2\geq A}} Z_{N_2}\zeta_0^{N_2} \prod_{i=3}^{M} Z_{N_i}\zeta_0^{N_i} \\
\leq\ & C(\tau_0) \sum_{N_2\geq A} Z_{N_2}\zeta_0^{N_2} \qquad (3.34)
\end{aligned}
$$

where $C(\tau_0)$ only depends on τ_0. Since Z_0 is finite the last expression goes to zero as $A \to \infty$. By estimating the remaining contribution to (3.32) and letting $A \to \infty$ it then follows as in [7] that

$$\nu_N(\{\tau \in \Gamma : B_R(\tau) = \tau_0\}) \to MW(\tau_0)Z_0^{M-1}\zeta_0^{|\tau_0|-M} \qquad (3.35)$$

as $N \to \infty$ which proves Property 2 for all ensembles which have a relation like in (3.30).

An important observation in these calculations is that when N gets larger all the mass tends to gather into one subtree attached to τ_0 and the sizes of the other subtrees are bounded by the constant A. In general this leaves two possibilities of how simply generated trees obeying (3.30) look like in the large N limit. Either the subtree with the large mass becomes an infinite branch as N goes to infinity, as always happens in the generic case, or the order of some of its vertices becomes infinite. There is some evidence from numerical calculations and analytical arguments that infinite vertices occur in a particular model for non-generic trees [14, 15]. This will be discussed in more detail later.

When the convergence of the measure has been established in [7] it is shown that the spectral dimension of the resulting infinite random graph is $d_s = 4/3$ and the Hausdorff dimension is $d_H = 2$. In the proof it is important that the trees have one infinite branch with identically and independently distributed critical Galton-Watson outgrowths with $f''(1) < \infty$, therefore obeying the relation (3.20).

3.4 The three phases

In the non-generic case $Z_0 = \rho$ as was explained above. Since all models with infinite ρ are generic we can take ρ to be finite when we study non-generic trees. In fact we can choose $\rho = 1$ without loss of generality by redefining the branching weights $w_n \to w_n \rho^{n-1}$. This redefinition does not change the probability distribution ν_N since

$$\prod_{i\in\tau\setminus r} w_{\sigma_i} \to \prod_{i\in\tau\setminus r} w_{\sigma_i}\rho^{\sigma_i - 1} = \rho^{N-1} \prod_{i\in\tau\setminus r} w_{\sigma_i}, \qquad \tau \in \Gamma_N \tag{3.36}$$

where we used that $\sum_{i\in\tau\setminus r} \sigma_i = 2N - 1$.

We start with a set of branching weights w_n which give $\rho = 1$ and at this stage the model can be either generic or non-generic. We fix the values of w_n for $n \geq 2$ but for now we let w_1 be a free parameter of the model. Define

$$h(Z) \equiv \frac{g(Z)}{Z}. \tag{3.37}$$

From (3.6) we see that $h(Z) = 1/\zeta(Z)$ for $Z \leq Z_0$. Differentiating h we get

$$h'(Z) = \frac{g(Z)}{Z^2}\left[Z\frac{g'(Z)}{g(Z)} - 1\right] \tag{3.38}$$

and again

$$h''(Z) = \frac{g''(Z)}{Z} - \frac{2}{Z}h'(Z). \tag{3.39}$$

The genericity condition can be interpreted as h having a minimum at $Z = Z_0 < 1$. For any $Z_0 < 1$ we can choose $w_1 = \sum_{n=2}^{\infty}(n-2)w_n Z_0^{n-1}$ making $Z_0\frac{g'(Z_0)}{g(Z_0)} = 1$. Then $h''(Z_0) = g''(Z_0)/Z_0 > 0$ which shows that the minimum is quadratic. Note that $Z_0\frac{g'(Z_0)}{g(Z_0)} = m$ where m is the mean number of offsprings defined in (3.14). We can clearly make any model with $\rho = 1$ generic by choosing

$$w_1 < \sum_{n=2}^{\infty}(n-2)w_n \equiv w_c \tag{3.40}$$

where w_c is a critical value for w_1 which depends on w_n for $n \geq 3$. It is interesting to note that the critical value is independent of w_2. Also note that if $w_c = \infty$, i.e. if $g'(z)$ diverges as $z \to 1$, we always have a generic ensemble.

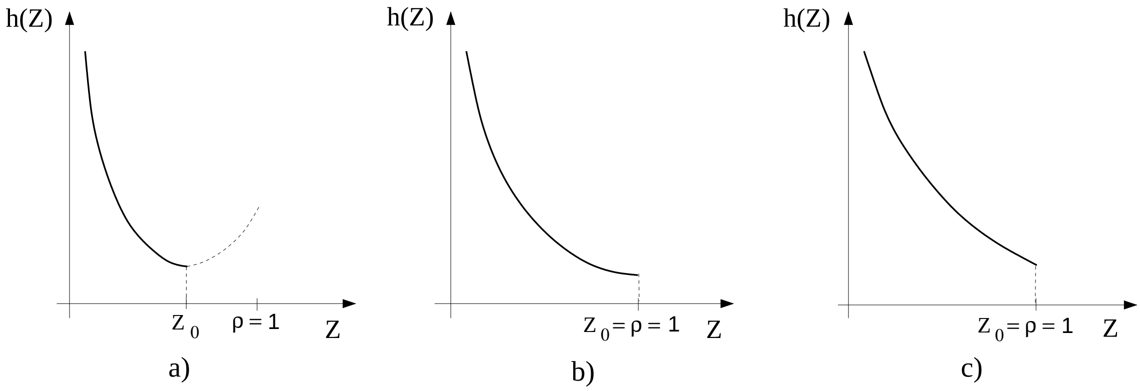

Figure 3.4: The three possible scenarios. a) Generic, quadratic minimum at Z_0. b) Critical, quadratic minimum at $Z_0 = \rho = 1$ if $g''(1) < \infty$. c) Subcritical, $h'(1) \neq 0$. The solid lines are also graphs of the function $1/\zeta(Z)$.

The next possible scenario is that h has a minimum at $Z = Z_0 = 1$. This happens when $w_1 = w_c$ or in other words when $m = \frac{g'(1)}{g(1)} = 1$. We see that although this is a non-generic ensemble, the trees are still critical Galton-Watson trees conditioned on the total size. They will be referred to as *critical trees*. We see that $h''(1) = g''(1) > 0$ which shows that the minimum is quadratic if $g''(1)$ is finite.

Finally, by choosing $w_1 > w_c$, h has no minimum and $m = \frac{g'(1)}{g(1)} < 1$. In this case the trees are non-generic, subcritical Galton-Watson trees conditioned on the total size. They will be referred to as *subcritical trees.*

To summarize, every model for which g has a finite radius of convergence has at most three phases. A generic phase when $w_1 < w_c$, a critical phase when $w_1 = w_c$ and a subcritical phase when $w_1 > w_c$. If $w_c = \infty$ there is only the generic phase.

3.5 A toy model

A simple model which has the properties in the previous section is the model $w_n = n^{-\beta}, \beta \in \mathbb{R}$ for $n \geq 2$ and $w_1 > 0$ free. It is clear that $\rho = 1$. This model has the advantage that it is possible to make explicit calculations. It has been studied in [14–16] both in the context of random trees and "balls in boxes" models.

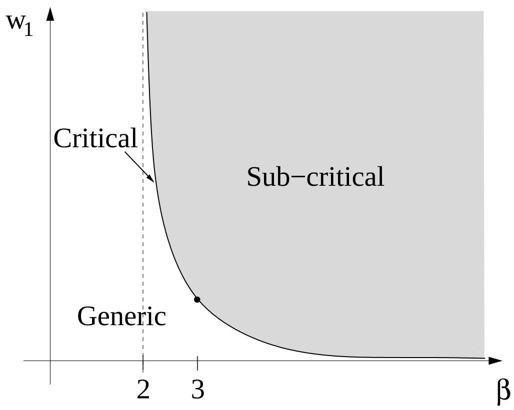

Figure 3.5: A diagram showing the three possible phases of trees. The critical line is determined by the equation $w_1 = w_c$.

We see right away that the case $\beta < 2$ is always generic since then $g'(z) \to \infty$ as $z \to 1$. The condition $w_1 = w_c$ gives a relation between w_1 and β which determines where the phase transition happens for any β. We call this relation the critical line and it is shown in Figure 3.5. Above the critical line we get subcritical trees but below it and to the left of it we get generic trees.

In [15] it is shown by expanding the function h around $Z = Z_0$ and inverting the expansion, that if there exists an exponent δ as in (3.30) it obeys

$$\delta = \begin{cases} \frac{3}{2} & \text{if } w_1 < w_c \text{ or if } w_1 = w_c \text{ and } \beta \geq 3 \\ \frac{\beta}{\beta-1} & \text{if } w_1 = w_c \text{ and } \beta \leq 3 \\ \beta & \text{if } w_1 > w_c. \end{cases} \tag{3.41}$$

The model can give any value of $\delta \geq 3/2$. We notice that on the critical line the value $\beta = 3$ plays a special role. It corresponds to the values of β, for which $g''(1)$ goes from being finite to being infinite. We will interpret this in the next section and discuss generalizations beyond the $n^{-\beta}$ model.

3.6 Critical trees

The critical value of w_1 which separates generic and subcritical trees is defined by $w_1 = w_c$. In the toy model in the previous section the exponent δ for critical trees agrees with the exponent for generic trees when $g''(1)$ is finite. The condition $g''(1) < \infty$ actually guarantees the generic behaviour (3.20) of critical Galton-Watson processes . After the convergence of the measure has been established in [7], this is in fact the only condition that is used to prove that the value of the spectral dimension of generic trees is $d_s = 4/3$. Therefore it is tempting to make the following conjecture.

> *Critical trees for which $g''(1)$ is finite share the properties of generic trees, having spectral dimension $d_s = 4/3$ and Hausdorff dimension $d_H = 2$.*

For now we will have to settle on the less general result in Theorem 2. First we prove the following lemma.

Lemma 6 *Consider critical trees which obey (3.30), (3.31) and $g''(1) < \infty$. Then $\delta = 3/2$.*

Proof: As was mentioned in the beginning of Section 3.4 the condition $g''(1) < \infty$ for critical trees implies that $h(Z)$ has a quadratic minimum at $Z = 1$. We can therefore do the following expansion

$$h(Z) - h(1) = \frac{h''(1)}{2}(1 - Z)^2 + o(1 - Z)^2. \tag{3.42}$$

By inverting this and remembering the definition of h we find that

$$Z(\zeta) = 1 - g(1)\sqrt{\frac{2}{g''(1)}}(\zeta_0 - \zeta)^{1/2} + o(\zeta_0 - \zeta)^{1/2}. \tag{3.43}$$

We now use Theorem 5 in chapter XIII.5 in [24] (a Tauberian theorem) to find that

$$\sum_{N=1}^{q} N Z_N \zeta_0^N \approx C q^{1/2} L(q) \tag{3.44}$$

where $C > 0$ is a constant and L is slowly varying. Since the trees obey (3.30) and (3.31) this shows that $\delta = 3/2$.

□

Theorem 2 *Consider critical trees which obey (3.30) and (3.31) . If* $\sum k^{5/2} w_{k+1} < \infty$ *the trees share the properties of generic trees, having spectral dimension* $d_s = 4/3$ *and Hausdorff dimension* $d_H = 2$.

Proof: To prove this it is enough to verify the convergence of the measure ν_N as explained above. Note that the condition in the theorem implies that $g''(1) < \infty$ and therefore the previous lemma shows that $\delta = 3/2$ and that Property 2 is true. All that is left is to prove is Property 1 and for later convenience we will do it for an arbitrary δ.

We show (3.28) by induction on R. The case $R = 1$ is trivial so we next consider the case $R = 2$. We can make the following estimate

$$\begin{aligned}
\nu_N(\{\tau \in \Gamma : |B_2(\tau)| = k+1\}) &= Z_N^{-1} w_{k+1} \sum_{N_1+\ldots+N_k=N-1} \prod_{i=1}^{k} Z_{N_i} \\
&\leq \zeta_0 k w_{k+1} \sum_{\substack{N_1+\ldots+N_k=N-1 \\ N_1 \geq (N-1)/k}} \frac{Z_{N_1}\zeta_0^{N_1}}{Z_N \zeta_0^N} \prod_{i=2}^{k} Z_{N_i} \zeta_0^{N_i} \\
&\leq C k w_{k+1} \Big(\frac{Nk}{N-1}\Big)^{\delta} \max_{N_1} \Big(\frac{L(N_1)}{L(N)}\Big) \sum_{N_2,\ldots,N_k \geq 1} \prod_{i=2}^{k} Z_{N_i}\zeta_0^{N_i} \\
&\leq C' k^{1+\delta} w_{k+1}
\end{aligned} \tag{3.45}$$

where $C, C' > 0$ are numbers independent of k and N. In the last step we used $Z_0 = 1$. Then we find

$$\nu_N(\{\tau \in \Gamma : |B_2(\tau)| > K\}) \leq C' \sum_{k=K}^{\infty} k^{1+\delta} w_{k+1}. \tag{3.46}$$

If this sum is finite it tends to zero as $K \to \infty$ uniformly in N proving the case $R = 2$.

Now assume that (3.28) holds for some $R \geq 2$. Since the set of balls $B_R(\tau)$ with volume at most K is finite for each fixed K it is enough to show that

$$\nu_N(\{\tau \in \Gamma : |B_{R+1}(\tau)| > K, B_R(\tau) = \tau_0\}) \to 0 \qquad \text{as} \qquad K \to \infty \tag{3.47}$$

uniformly in N for every finite tree τ_0 of height R. With a slight generalization of the arguments in [7] we can show that

$$\begin{aligned} &\nu_N(\{\tau \in \Gamma : |B_{R+1}(\tau)| > K, B_R(\tau) = \tau_0\}) \\ \leq\ & C'' \left(\sum_{k=1}^{\infty} k^{1+\delta} w_{k+1} \right)^{M-1} \left(\sum_{k > (K - |\tau_0|)/M} k^{1+\delta} w_{k+1} \right) \end{aligned} \tag{3.48}$$

where $C'' > 0$ only depends on τ_0 and M is the number of vertices in τ_0 at distance R from the root. This goes to zero uniformly in N as $K \to \infty$ if the last two sums are finite.

□

The case $g''(1) = \infty$ for critical trees is more difficult to treat. It is not possible to show the convergence of the measure with the direct approach used here and in [7]. If the convergence could be established it would be possible to find the spectral dimension for some specific models like (3.21) . This model actually includes the case $w_n = n^{-\beta}, w_1 = w_c$ with $2 < \beta < 3$ and L constant. Then (3.23) becomes

$$\mu(\{\tau \in \Gamma | h(T) > R\}) \approx R^{\frac{1}{\beta - 2}}. \tag{3.49}$$

By assuming the existence of the measure and using this relation, a direct application of the methods of [7] gives a lower bound on the spectral dimension

$$d_s \geq 2\frac{\beta - 1}{2\beta - 3}. \tag{3.50}$$

This lower bound is the same as the claimed exact value of the spectral dimension in [5, 16]. It is not possible to find an upper bound with the methods of [7] since they rely on $g''(1) < \infty$.

3.7 Subcritical trees

For now there are no rigorous results on the limiting behaviour of the measure on subcritical trees. In this section we will however give some arguments for the large size behaviour of subcritical trees which allow us to cook up a possible limiting measure.

To begin with we consider only the model of Section 3.5. In the relation (3.41) we see that for subcritical trees $\delta = \beta$. In this case $Z_N \zeta_0^N$ behaves exactly like w_n. When we try to prove the convergence of the measure the proof of Property 1 goes seriously wrong since $k^{1+\delta-\beta} = k$ and the sum of this never converges. Although this is of course no disproof of Property 1 this exact cancellation between β and δ indicates a different behaviour. It is in fact natural to expect a dramatically different limiting behaviour because subcritical trees correspond to subcritical Galton-Watson processes. This seems to hold even beyond the $n^{-\beta}$ model. If we for example let $w_n = e^{-\sqrt{n}}$ and $w_1 > w_c$ then by repeated differentiation of (3.6) we see that $Z_N \zeta_0^N$ falls faster than any power of N, and in that way behaves similar to w_n .

As was explained in Section 3.3 there are two possible scenarios as the tree size grows large. Either there emerges exactly one infinite branch with finite outgrowths or one or more vertices of infinite order appear. We will from now on refer to vertices of infinite order as traps. If Property 1 is in fact not true in the subcritical case we expect traps to occur. It is argued in [14, 15] with numerical calculations and analytical arguments that exactly one trap occurs . For large finite N its size is estimated to be $(1-m)N$ where $m = \frac{g'(1)}{g(1)} < 1$ is the mean value of offsprings of the subcritical Galton-Watson process. We cannot prove this but we can check if this is consistent with our picture of subcritical trees being size conditioned subcritical Galton-Watson trees.

First, observe the behaviour of the mean size of unconditioned subcritical Galton-Watson trees (3.18) with m. The mean size $\langle |B_\infty| \rangle_\mu$ is always finite so it is not impossible to imagine that by conditioning the Galton-Watson process on very large trees of fixed size, the limiting distribution would have trees of bounded height. This would indicate the occurrence of a trap. When m is small, $\langle |B_\infty| \rangle_\mu$ is small and so the trees are crumpled. Therefore it is natural to expect the trap size to increase as predicted. When $m \to 1$ the mean size goes to infinity and the trees become longer and are stretched towards the critical case. The trap size would then go to zero as predicted.

Secondly, we can prove that there can occur at most one trap. We can estimate the probability that there exist two vertices, i and j such that $\sigma(i) \geq \epsilon_i N$ and $\sigma(j) \geq \epsilon_j N$, $\epsilon_i, \epsilon_j > 0$. We draw the trees as in Figure 3.6 where we assume that the order of the vertex j is p. Each balloon $N_1, \ldots, N_{p-1}$ along with the link to j is a tree of size $N_k + 1$ with root j. The balloon labelled with N_p along with the link to j is a tree of size $N_p + 1$ with root r and one marked vertex j of order one. The partition function for the balloon with the marked vertex is $\partial Z_{N_p+1}/\partial w_1$ because we can choose the marked vertex in $e(\tau)$ ways where $e(\tau)$ is the number of vertices in τ of order one (excluding the root). It is easy to convince oneself that

$$w_1 \frac{\partial Z_{N+1}}{\partial w_1} \leq N Z_{N+1}. \tag{3.51}$$

The partition function for each of the other balloons is Z_{N_k+1}. Finally the weight of the vertex j is w_p. We get the following estimate by summing over all these configurations

$$\begin{aligned}
& \nu_N(\{\tau \in \Gamma_N | \exists i, j \in \tau \ \text{ such that } \ \sigma(j) \geq \epsilon_j N \ \text{ and } \ \sigma(i) \geq \epsilon_i N\}) \\
\leq \ & Z_N^{-1} \sum_{N \geq p \geq \epsilon_j N} w_p \Big[(p-1) \sum_{\substack{N_1 + \ldots + N_p = N-p \\ N_1 \geq \epsilon_i N}} \frac{\partial Z_{N_p+1}}{\partial w_1} \prod_{k=1}^{p-1} Z_{N_k+1} + \sum_{\substack{N_1 + \ldots + N_p = N-p \\ N_p \geq \epsilon_i N}} \frac{\partial Z_{N_p+1}}{\partial w_1} \prod_{k=1}^{p-1} Z_{N_k+1} \Big] \\
\leq \ & C_{\epsilon_j} N \underbrace{Z_N^{-1} w_N \zeta_0^{-N}}_{< const.} \sum_{N \geq p \geq \epsilon_j N} \sum_{\substack{N_1 + \ldots + N_p = N-p \\ N_1 \geq \epsilon_i N}} (N_p + 1) \prod_{k=1}^{p} Z_{N_k+1} \zeta_0^{N_k+1} \\
\leq \ & C'_{\epsilon_j} N \sum_{N \geq p \geq \epsilon_j N} \sum_{N_1 \geq \epsilon_i N} \left[\frac{N - 1 - N_1}{p-1} \right] \underbrace{Z_{N_1+1} \zeta_0^{N_1+1}}_{\sim N_1^{-\delta}} \sum_{N_2 + \ldots + N_p = N-p-N_1} \prod_{k=2}^{p} Z_{N_k+1} \zeta_0^{N_k+1} \\
\leq \ & C_{\epsilon_i, \epsilon_j} N^{1-\delta} \sum_{N \geq p \geq \epsilon_j N} \left(\sum_{n=0}^{\infty} Z_{n+1} \zeta_0^{n+1} \right)^{p-1} \leq C'_{\epsilon_i, \epsilon_j} N^{2-\delta}.
\end{aligned}$$

Here, $C_{(\cdot)}$ and $C'_{(\cdot)}$ are positive numbers which only depend on their subscripts. Since $\delta > 2$ the last expression goes to zero as $N \to \infty$. For convenience we left the slowly varying function out of these calculation but it enters as in (3.45) and can be estimated as before.

3.7.1 A candidate for a limiting measure

From the above arguments we are ready to make an educated guess on what the limiting measure on subcritical trees might look like. We assume that exactly one trap emerges when the size goes to infinity. We also assume that the trap takes care of the size constraint on the conditioned Galton-Watson process just as the infinite branch did in the generic case. Therefore the rest of the graph grows like an unconditioned, subcritical Galton-Watson process.

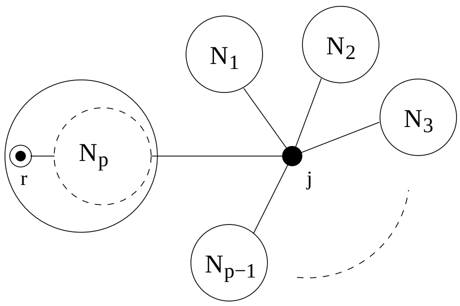

Figure 3.6: A graph with a vertex j of large order p and another vertex of large order inside one of the balloons.

By looking at Figure 3.6 we can imagine the vertex j to be the trap, the balloons labelled with $N_1, \ldots, N_{p-1}$ to be subcritical Galton-Watson trees with root j and the balloon labelled with N_p to be a subcritical Galton-Watson tree with root r and one marked vertex j (the trap) of order one. We assume that each balloon grows independent of the others. We know the probability measure for the balloons with unmarked vertices, it is simply μ defined in (3.10) and (3.12). To check for consistency, note that the expectation value of the size of each of the balloons, when N is large, is approximately $\langle |B_\infty| \rangle_\mu = 1/(1-m)$ and the expected number of balloons is $(1-m)N$ (the order of j). These two numbers multiplied together, give the total size N which shows consistency.

We denote the probability measure for rooted Galton-Watson trees with one trap with μ^*. It can be constructed from μ by noticing that the probability for each tree to occur is the same as before

$$\mu^*(\tau) = D\zeta_0^{|\tau|-1} \prod_{i \in \tau \setminus \{r,j\}} w_{\sigma(i)} \tag{3.52}$$

but we exclude the weight of the trap and there is a different normalization constant

D. To find the normalization constant we note that for each unmarked tree τ we get $e(\tau)$ marked trees where $e(\tau)$ denotes the number of vertices in τ of order one (excluding the root). Therefore

$$\frac{1}{D} = \sum_{N=1}^{\infty} \zeta_0^{N-1} \sum_{\tau \in \Gamma_N} e(\tau) \prod_{i \in \tau \setminus \{r,j\}} w_{\sigma(i)} = \frac{1}{\zeta_0} \sum_{N=1}^{\infty} \frac{\partial Z_N}{\partial w_1} \zeta_0^N = \frac{1}{1-m} \tag{3.53}$$

where we found the last step by differentiating (3.6) with respect to w_1 and using $Z_0 = 1$. Therefore $D = 1 - m$.

We can look at the measure μ^* in the following way. Each tree for which the shortest path between the root and the trap equals h can be drawn as in Figure 3.7.

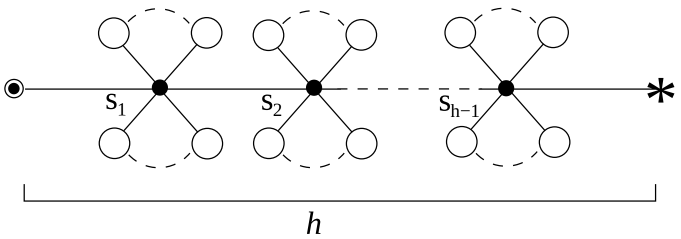

Figure 3.7: A possible description of infinite, subcritical trees. The tree has the graph M_h as a base with the probability $p(h)$ given in (3.54) and it has finite sub-critical Galton-Watson outgrowths. The trap is denoted with an asterisk.

We call the linear subgraph which starts at the root and ends at the trap M_h. We denote the vertex with graph distance i from the root on the subgraph M_h with s_i. Each balloon attached to one of the vertices s_i, grows independently according to μ. The probability distribution of the length h is

$$\begin{aligned} p(h) &= D\Big(\sum_{k,l \geq 0} \zeta_0 w_{2+k+l}\Big)^{h-1} = (1-m)g(1)^{-h+1}\Big(\sum_{n=0}^{\infty}(n+1)w_{2+n}\Big)^{h-1} \\ &= (1-m)g(1)^{-h+1}g'(1)^{h-1} = (1-m)m^{h-1}. \end{aligned} \tag{3.54}$$

Note that the probabilities $p(h)$ sum to one. Since $m < 1$, $p(h)$ decays exponentially which means that the probability of the trap being close to the root is relatively high. The conditional probability of having k left branches and l right branches attached to a vertex s_i given that M_h is a subgraph of the tree is

$$\phi(k,l) = \frac{1}{m}\zeta_0 w_{2+k+l} \tag{3.55}$$

and is identical for each vertex.

To summarize, a possible description of infinite subcritical trees is the following. There occurs exactly one trap and its distance from the root, h, is distributed by $p(h)$. Subcritical Galton-Watson trees grow from the subgraph M_h according to ϕ and μ. The trap has infinitely many subcritical Galton-Watson trees growing from it distributed by μ. We will not worry about the outgrowths from the trap since a random walk which hits the trap will never return back to the root and balls centered on the root with radius greater than the distance to the trap have infinite volume. This means that the trap outgrowths neither affect the spectral dimension nor the Hausdorff dimension.

We would like to say something about the spectral and Hausdorff dimension of the above random tree. First we consider some simple random tree models which are related to the subcritical trees.

3.7.2 Examples of random trees with one trap

Consider the graph M_l mentioned in the previous section. It looks like N_l but it

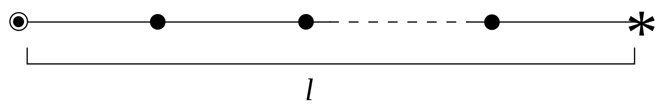

Figure 3.8: The graph M_l with a trap denoted with an asterisk.

has a trap at the opposite end of the root. If a random walk hits the trap we say that it returns to the root with probability zero. For a fixed graph M_l it is therefore obvious that the spectral dimension is infinite because the random walk eventually goes to the trap with probability one. If we define the trap to have infinite volume the graph also has an infinite Hausdorff dimension. This seems like the end of the story but it turns out that we can get a finite spectral dimension by considering a random graph where we put a probability distribution on the length l and make sure that the trap has a high probability of being far from the root. The Hausdorff dimension is however always infinite.

To find the first return generating function of M_l we use the recurrence relation in (2.12) replacing N_l with M_l but with a different boundary condition $P_{M_1}(x) = 0$. To solve this we use the methods of Appendix A in [6]. The result is very similar to the result for N_l in (2.13)

$$P_{M_l}(x) = 1 - \sqrt{x}\frac{(1+\sqrt{x})^l + (1-\sqrt{x})^l}{(1+\sqrt{x})^l - (1-\sqrt{x})^l}. \tag{3.56}$$

The square root in this formula is actually deceiving because P_{M_l} is in fact a rational function for all l. By expanding the brackets using the binomial formula we can write the corresponding return probability generating function as

$$Q_{M_l}(x) = \frac{R_l(x)}{S_l(x)} \tag{3.57}$$

where R_l and S_l are the polynomials

$$R_l(x) = \sum_{i=0}^{\left[\frac{l-1}{2}\right]} \binom{l}{2i+1} x^i \qquad \text{and} \qquad S_l(x) = \sum_{i=0}^{\left[\frac{l}{2}\right]} \binom{l}{2i} x^i. \tag{3.58}$$

From these expressions one can see that $Q^{(n)}_{M_l}(0)$ is a polynomial in l of degree $2n+1$. In particular $Q^{(n)}_{M_l}(0)$ is finite for all l showing that the spectral dimension is indeed infinite for a fixed l. Now, pick a probability distribution $p_l = cl^{-a}$ on the set $\{M_l | l \geq 1\}$ and define a return generating function for the corresponding random graph

$$\bar{Q}(x) = \sum_{l=1}^{\infty} p_l Q_{M_l}(x). \tag{3.59}$$

The convergence or divergence of this sum or its derivatives can now be determined by inserting $x = 0$ and finding the highest exponent of l. From that we can conclude that if the graph has a spectral dimension d_s it obeys $a - 2 \leq d_s \leq a + 2$. In the case when $1 < a \leq 2$ it is in fact easy to show by comparing the sum (3.59) with an integral that $d_s = a$. This relation probably holds for higher values of a but it becomes messier to confirm since it involves taking higher and higher derivatives of Q_{M_l}.

These arguments show that we can get a finite spectral dimension for the random graph in whatever range we like. The bounds on d_s are like we expected. By decreasing a the probability of having the trap close to the root decreases and the spectral dimension is lowered. We note that if p_l decreases faster than any power of l then the spectral dimension is always infinite. In the subcritical random trees this probability decreases exponentially which implies that they might have an infinite spectral dimension. But the graph M_l has no branches and it turns out that it approximates the subcritical trees poorly.

We look at another model of a tree where we attach q single links to each vertex of M_l as is shown in Figure 3.9. We call the resulting graph $M_{l;q}$. Let's call the graph which is made of the bundle of q single links F_q and let the vertex of order q be the root. The first return generating function for F_q is

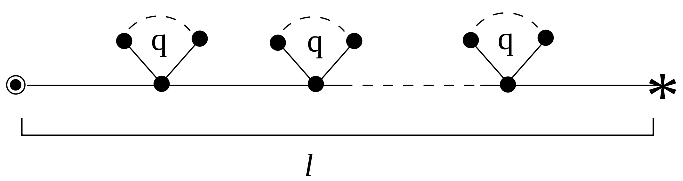

Figure 3.9: The graph $M_{l;q}$ made by attaching a graph F_q to each vertex of M_l except the root.

$$P_{F_q}(x) = 1 - x. \tag{3.60}$$

We can use the methods of Section 2.2 to find the first return generating function for $M_{l;q}$. The function K_{G_1,G_2} in (2.24) is simply

$$K_{M_{l;q},F_q}(x;\omega) = \Big(\frac{2}{2+qx}\Big)^{|\omega|-1} \tag{3.61}$$

so the first return generating function becomes

$$P_{M_{l;q}}(x) = (1 + \frac{q}{2}x)P_{M_l}(x_q(x)) \tag{3.62}$$

where we defined

$$x_q(x) = \frac{\frac{q^2}{4}x^2 + (1+q)x}{(1+\frac{q}{2}x)^2}. \tag{3.63}$$

We see that $x_q(0) = 0$. Repeated differentiation of $x_q(x)$ shows that $x_q^{(n)}(0)$ is a polynomial in q of degree n. Therefore, repeated differentiation of $Q_{M_{l;q}}(x)$ shows that $Q^{(n)}_{M_{l;q}}(0)$ is a polynomial in l of degree $2n+1$ and in q of degree n. For any fixed graph of this kind the spectral dimension is therefore infinite.

Now let's make both q and l random according to some distributions r_q and p_l respectively. Since

$$Q^{(n)}_{M_{l;q}}(0) = A_n l^{2n+1} q^n \ + \ \text{ lower powers of } l \text{ and } q \qquad A_n \neq 0 \text{ constant} \tag{3.64}$$

we can clearly make any derivative of the average of $Q_{M_{l;q}}(x)$ diverge as $x \to 0$ by tuning the probabilities. This gives a finite spectral dimension which depends on both distributions. What is more interesting is that for any p_l we can make any derivative diverge with a suitable choice of r_q. This means that even though p_l drops exponentially the branches attached to M_l can slow the random walker down so that it has little probability of meeting the trap which results in a finite spectral dimension. This effect might give us a finite spectral dimension of subcritical trees.

By attaching graphs more complicated than F_q to M_l we can slow the random walker even further down. For example, consider a rooted tree which has a root of order one and a single vertex of order q_2. Attach q_1 copies of it to every vertex of M_l except the root. Then with the same analysis as above we find that

$$Q^{(n)}_{M_{l;q_1 q_2}}(0) = B_n l^{2n+1} q_1^n q_2^n \ + \ \text{ lower powers of } l \text{ and } q \qquad B_n \neq 0 \text{ constant} \tag{3.65}$$

where we have denoted the resulting graph with $M_{l;q_1q_2}$. If we distribute q_1 and q_2 independently with the same probability distribution we have a very similar situation as in (3.64). However, if we for example put $q_1 = q_2 \equiv q$ and put the probability distribution on q we make the resulting random graph less transient.

3.7.3 Dimensions of the subcritical limiting measure

To conclude we would like to say something about the spectral and Hausdorff dimension of the proposed subcritical random tree. First of all we can right away deduce that the Hausdorff dimension is $d_H = \infty$ since there is a nonzero probability of having the trap at a finite distance from the root.

The spectral dimension could however be finite even though $p(h)$ drops exponentially, since the branches attached to M_h could serve to slow the random walker down on its way to the trap. This needs to be carefully checked.

Let $\bar{P}^*(x)$ and $\bar{Q}^*(x)$ be the first return and return probability generating functions averaged with respect to the measure μ^*. We construct graphs $M_{h;q}$ like in the previous section and compare their return probability generating function to $\bar{P}^*(x)$ and $\bar{Q}^*(x)$ using the monotonicity lemma from Chapter 2. First consider a subcritical random tree. The probability that a vertex has at least q branches is

$$a(q) = \sum_{k+l \geq q} \phi(k,l). \tag{3.66}$$

The probability that the number of branches of each vertex of M_h is at least q is then

$$b_h(q) = a(q)^{h-1}. \tag{3.67}$$

Then the probability that there are exactly q branches at some vertex of M_h and at least q branches at all the other vertices is

$$c_h(q) = b_h(q) - b_h(q+1). \tag{3.68}$$

Let Γ^h be the set of trees which have M_h as a subgraph and let $\Gamma^{h;q}$ be the set of trees which have $M_{h;q}$ as a subgraph and at least one vertex on M_h of order q. We then define

$$\bar{Q}_h(x) = \int_{\tau \in \Gamma^h} Q_\tau(x) d\mu^*(\tau|\Gamma^h) \tag{3.69}$$

as the return probability generating function averaged over the branches of M_h and

$$\bar{Q}_{h;q}(x) = \int_{\tau \in \Gamma^{h;q}} Q_\tau(x) d\mu^*(\tau|\Gamma^{h;q}) \tag{3.70}$$

as the return probability generating function averaged over the branches of $M_{h;q}$. We define $\bar{P}_h(x)$ and $\bar{P}_{h;q}(x)$ in the same way. We can then write

$$\bar{Q}^*(x) = \sum_{h=1}^{\infty} p(h)\bar{Q}_h(x) = \sum_{q=0}^{\infty}\sum_{h=1}^{\infty} p(h)c_h(q)\bar{Q}_{h;q}(x) \tag{3.71}$$

and

$$\bar{P}^*(x) = \sum_{h=1}^{\infty} p(h)\bar{P}_h(x) = \sum_{q=0}^{\infty}\sum_{h=1}^{\infty} p(h)c_h(q)\bar{P}_{h;q}(x). \tag{3.72}$$

We start by finding a lower bound on the spectral dimension. Let $\bar{Q}(x)$ be the same as in (3.59) with $p_h = p(h)$ from (3.54). By the monotonicity lemma of Chapter 2 we have $\bar{Q}_h(x) \leq Q_{M_h}(x)$ for all h. Therefore, by (3.71) $\bar{Q}^*(x) \leq \bar{Q}(x)$. Now $\bar{Q}(0)$

is finite because of the choice of p_h and therefore $\bar{Q}^*(0)$ is finite and so the random tree is transient and $d_s \geq 2$. This bound is model independent and it is not clear if it is optimal.

Now we would like to find an upper bound. Since we have just shown that subcritical random trees are transient it is clear that we have to compare derivatives of return generating functions. We therefore use the methods of Section 2.2.1. Define

$$\bar{H}_{h;q}(x;n) = \int_{\tau\in\Gamma^{h;q}} H_{\tau,M_{h;q}}(x;n)d\mu^*(\tau|\Gamma^{h;q}) \tag{3.73}$$

where the integrand has the same definition as in Section 2.2.1 and n is chosen such that $(-1)^{n-1}\bar{P}^{*\ (n-1)}(0) < \infty$. An example of such an n is $n = 1$ but we will make the choice more optimal later. With the same methods as in (2.27) we get

$$(-1)^n \bar{H}'_{h;q}(x;n) \leq (-1)^n \bar{P}^{(n)}_{h;q}(x). \tag{3.74}$$

We have $(-1)^{(n-1)}\bar{H}_{h;q}(x;n) \leq (-1)^{(n-1)}P^{(n-1)}_{M_{h;q}}(x)$ with equality in $x = 0$. Therefore by using the generalized mean value theorem as before, we find that for every $x \in]0,1[$ there exists a $\xi \in]0,x[$ such that

$$(-1)^n \bar{P}^{(n)}_{h;q}(\xi) \geq (-1)^n P^{(n)}_{M_{h;q}}(\xi). \tag{3.75}$$

By summing over h and q we get

$$(-1)^n \bar{P}^{*\ (n)}(\xi) \geq (-1)^n \sum_{q=0}^{\infty}\sum_{h=1}^{\infty} p(h)c_h(q)P^{(n)}_{M_{h;q}}(\xi). \tag{3.76}$$

We would like to find if and how this diverges when $\xi \to 0$ to get an upper bound on the spectral dimension. We start by throwing away every term of the sum over h except $h = 2$. From (3.62) and (3.63) we find

$$P_{M_{2,q}}(x) = \frac{1-x}{2+qx} \tag{3.77}$$

and it is easily proved by induction that

$$P^{(n)}_{M_{2,q}}(x) = (-1)^n n!\frac{q^{n-1}(q+2)}{(2+qx)^{n+1}}, \qquad \text{for } n \geq 1. \tag{3.78}$$

Therefore

$$
\begin{aligned}
(-1)^n \bar{P}^{*\ (n)}(\xi) &\geq (1-m)\zeta_0 n! \sum_{q=0}^{\infty} \frac{w_{q+1} q^{n-1}(q+1)(q+2)}{(2+q\xi)^{n+1}} && (3.79) \\
&= (1-m)\zeta_0 n! \sum_{q=0}^{\infty} \frac{q^{n-1}(q+1)^{-\beta+1}(q+2)}{(2+q\xi)^{n+1}} && (3.80)
\end{aligned}
$$

Note that for every $\xi > 0$ this sum converges but if we put $\xi = 0$ it diverges given that $\beta \leq n+2$. We are now ready to make an optimal choice of n. Choose n such that $n+1 < \beta \leq n+2$. Consider first the possibility that $(-1)^{n-1}\bar{P}^{*\ (n-1)}(0) = \infty$. In this case $d_s \leq 2n < 2(\beta-1)$. If however $(-1)^{n-1}\bar{P}^{*\ (n-1)}(0) < \infty$ the choice of n fullfills the condition explained below (3.73) and we can therefore use the estimate in (3.80). We then compare the last sum in (3.80) to an integral and get

$$
\begin{aligned}
\sum_{q=0}^{\infty} \frac{q^{n-1}(q+1)^{-\beta+1}(q+2)}{(2+q\xi)^{n+1}} &\approx C\int_1^{\infty} \frac{t^{n+1-\beta}}{(2+t\xi)^{n+1}} dt, \qquad \text{as} \quad \xi \to 0 \\
&= C\xi^{\beta-n-2} \int_{\xi}^{\infty} \frac{y^{n+1-\beta}}{(2+y)^{n+1}} dy && (3.81)
\end{aligned}
$$

where $C > 0$ is a constant. If $\beta < n+2$ the last integral is convergent even when $\xi \to 0$. In this case

$$
(-1)^n \bar{P}^{*\ (n)}(\xi) \geq C' \xi^{\beta-n-2} \tag{3.82}
$$

where $C' > 0$ is independent of ξ. Then we see from (2.11) that the spectral dimension has an upper bound $d_s \leq 2(\beta-1)$. If however $\beta = n+2$ the singular behaviour in front of the integral disappears and we have to perform the integral. We find that it diverges like a logarithm as $\xi \to 0$ and we get the same upper bound as before.

We have therefore shown that the spectral dimension of the subcritical trees is in the interval $2 \leq d_s \leq 2(\beta-1)$ for all $\beta > 2$. It is proposed in [5] that the spectral dimension of subcritical trees equals 2 which is clearly in this range.

In all of the above discussion we have worked with $w_n = n^{-\beta}, w_1 > w_c$. If we assume that the measure only depends on the trees being subcritical then we could for example take $w_n \sim n^{-\beta}, w_1 > w_c$. Then (3.79) shows that we get the same bounds on the spectral dimension as before.

4

Conclusions

In Chapter 2 we found inequalities between first return generating functions of bundles structures, generalizing the monotonicity lemmas in [6, 7]. These monotonicity results allowed us to investigate how the spectral dimension of a graph changes when recurrent graphs are attached to it. It turns out that recurrent graphs stay recurrent and their spectral dimension increases but transient graphs stay transient and their spectral dimension decreases. This gives bounds on the spectral dimension of random brushes and in the same way it is possible to deal with general random bundled structures with a fixed base and random fibers. Similar methods were used to find bounds on the spectral dimension of the conjectured limiting measure on subcritical trees. This case was slightly different since the transient base M_h was also random.

The methods used here to study the spectral dimension only work for bundled structures for which the diffusion properties of the base and the fibers are known. It would be interesting to understand the properties of the spectral dimension of more general graphs.

In Chapter 3, non-generic trees were studied. They were divided into two categories, critical trees and subcritical trees, depending on weather they are related to critical or subcritical Galton-Watson processes. It was shown that in any model for which the generating function of the branching weights, g, has a finite radius of convergence the trees can have three phases: generic, critical and subcritical depending on the weight of the leaves w_1.

A conjecture was made, that all critical trees for which $g''(1) < \infty$ share the properties of generic trees. A less general statement was proved and some possible results in the case $g''(1) = \infty$ were discussed. We might need fancier methods to deal with critical trees, than the straight forward estimates used to prove Properties 1 and 2. One idea is to use saddle point methods to approximate the sums encountered, in the large N limit. However, saddle point methods usually rely on conditions similar to the genericity condition in the tree model.

A conjecture was made on a limiting measure for subcritical trees using arguments from [14–16]. The measure is concentrated on trees with exactly one trap with infinitely many finite subcritical Galton-Watson outgrowths, one of them containing the root. One idea to prove the existence of this measure could be to show that the ν_N probability that a tree has height greater than h goes to zero, uniformly in N as $h \to \infty$. The measure was shown to have Hausdorff dimension $d_H = \infty$ and spectral dimension $d_s \geq 2$ with a model dependent upper bound. The measure is interesting in itself, even though it would turn out to be the wrong limiting measure, since it is an example of a random graph with a trap which can still have a finite spectral dimension.

Bibliography

[1] D. ben Avraham and S. Havlin, *Diffusion and reactions in fractals and disordered systems.* Cambridge university press, Cambridge, 2000.

[2] J. Ambjørn, B. Durhuus, and T. Jonsson, *Quantum geometry: a statistical field theory approach.* Cambridge university press, Cambridge, 1997.

[3] S. Alexander and R. Orbach, *Density of states of fractals: 'fractons'*, *J. Phys. Lett.* **43** (1982) 625–631.

[4] T. Jonsson and J. F. Wheater, *The spectral dimension of the branched polymer phase of two-dimensional quantum gravity*, *Nucl. Phys.* **B515** (1998) 549–574, [hep-lat/9710024].

[5] J. D. Correia and J. F. Wheater, *The spectral dimension of non-generic branched polymer ensembles*, *Phys. Lett.* **B422** (1998) 76–81, [hep-th/9712058].

[6] B. Durhuus, T. Jonsson, and J. F. Wheater, *Random walks on combs*, *J. Phys.* **A39** (2006) 1009–1038, [hep-th/0509191].

[7] B. Durhuus, T. Jonsson, and J. F. Wheater, *The spectral dimension of generic trees*, math-ph/0607020.

[8] D. Cassi and S. Regina, *Random walks on bundled structures*, *Phys. Rev. Lett.* **76** (1996) 2914–2917.

[9] J. Ambjørn, J. Jurkiewicz, and Y. Watabiki, *On the fractal structure of two-dimensional quantum gravity*, *Nucl. Phys.* **B454** (1995) 313–342, [hep-lat/9507014].

[10] J. Ambjørn, K. N. Anagnostopoulos, T. Ichihara, L. Jensen, and Y. Watabiki, *Quantum geometry and diffusion*, *JHEP* **11** (1998) 022, [hep-lat/9808027].

[11] J. Ambjørn, D. Boulatov, J. L. Nielsen, J. Rolf, and Y. Watabiki, *The spectral dimension of 2d quantum gravity*, *JHEP* **02** (1998) 010, [hep-th/9801099].

[12] J. Ambjørn, J. Jurkiewicz, and R. Loll, *Spectral dimension of the universe*, *Phys. Rev. Lett.* **95** (2005) 171301, [hep-th/0505113].

[13] J. Ambjørn, J. Jurkiewicz, and R. Loll, *Reconstructing the universe*, *Phys. Rev.* **D72** (2005) 064014, [hep-th/0505154].

[14] P. Bialas, Z. Burda, and D. Johnston, *Condensation in the backgammon model*, *Nucl. Phys.* **B493** (1997) 505–516, [cond-mat/9609264].

[15] P. Bialas and Z. Burda, *Phase transition in fluctuating branched geometry*, *Phys. Lett.* **B384** (1996) 75–80, [hep-lat/9605020].

[16] Z. Burda, J. D. Correia, and A. Krzywicki, *Statistical ensemble of scale-free random graphs*, *Phys. Rev.* **E64** (2001) 046118, [cond-mat/0104155].

[17] R. A. Adams, *Calculus, a complete course*, vol. 5. Pearson Ed., Canada, 2003.

[18] B. Durhuus, *Probabilistic Aspects of Infinite Trees and Surfaces*, *Acta Physica Polonica B* **34** (Oct., 2003) 4795.

[19] H. S. Wilf, *Generatingfunctionology.* A. K. Peters, Ltd., Natick, MA, USA, 2006.

[20] K. Athreya and P. Ney, *Branching Processes.* Dover Publications, Inc, New York, 1972.

[21] T. E. Harris, *The theory of branching processes.* Springer-Verlag, Berlin, 1963.

[22] R. Slack, *A branching process with mean one and possibly infinite variance*, *Z. Wahrsche* **9** (1968) 139–145.

[23] P. Flajolet and R. Sedgewick, *Analytic combinatorics.* Online book available at http://algo.inria.fr/flajolet/Publications/books.html.

[24] W. Feller, *An introduction to probability theory and its applications*, vol. 2. Wiley, New York, 1966.